

ELECTRODYNAMICS:
basic course
and
applications in condensed matter

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The present one-semester course was given for the PhD students of the Federal University of Minas Gerais (UFMG), Brazil, in 1998. The principal objectives of course was, despite the introduction to the classical Electrodynamics, to give the review of the contemporary developments of the condensed matter physics. Substantial attention was given to solution of the problems and to introduction to the related technique of the theoretical physics (vector and tensor analysis, theory of analytic functions, methods of statistical physics etc.). The written by students and attached as appendix chapters concern the modern applications of Electrodynamics.

The major part of the course was based on the classical books:

L. D. Landau and E. M. Lifshitz, *The classical Theory of Fields*, II, Addison-Wesly Press, Pergamon Press-1951

L. D. Landau and E. M. Lifshitz, *Electrodynamics of continuous media*, Vol. VIII, Pergamon Press-1960

J. D. Jackson, *Classical Electrodynamics*, J. Wiley & Sons, 1975

N. W. Ashcroft and N. Mermin *Solid State Physics*,
Ch. Kittel, *Introduction to Solid State Physics*

and on the traditional course of theoretical physics of the Moscow Physico-Technical University, that was kindly given by E. Scherman.

O presente curso de um semestre foi oferecido para estudantes de doutorado da Universidade Federal de Minas Gerais (UFMG), Brasil, em 1998. Os principais objetivos do curso foram, além da introdução à eletrodinâmica clássica, dar uma revisão dos desenvolvimentos recentes da física da matéria condensada. Substantial atenção foi dada para a solução de problemas e para introduzir técnicas da física teórica (análise vetorial e tensorial, teoria de funções analíticas, métodos da física estatística etc.). Os textos sobre aplicações modernas da eletrodinâmica, escritos pelos estudantes, foram acrescentados como apêndices.

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I. VECTOR ALGEBRA AND ANALYSIS

A. Vector algebra:

Scalar product:

$$(\mathbf{a} \cdot \mathbf{b}) = ab \cos \alpha = a_x b_x + a_y b_y + a_z b_z$$

(other notations: $(\mathbf{a}, \mathbf{b}), \mathbf{a} \cdot \mathbf{b}$)

properties: $(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{b} \cdot \mathbf{a}), (\mathbf{a} \cdot \mathbf{a}) = a^2, (\mathbf{a} \cdot \mathbf{b}) = 0$ if

$\mathbf{a} \perp \mathbf{b}$.

Vector product: $[\mathbf{a} \times \mathbf{b}] =$

$$= \mathbf{i}(a_y b_z - a_z b_y) + \mathbf{j}(a_z b_x - a_x b_z) + \mathbf{k}(a_x b_y - a_y b_x)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

(other notations: $[\mathbf{a}, \mathbf{b}], \mathbf{a} \times \mathbf{b}, \mathbf{a} \wedge \mathbf{b}$)

properties: $[\mathbf{a} \times \mathbf{b}] = -[\mathbf{b} \times \mathbf{a}], [\mathbf{a} \times \mathbf{a}] = 0$

Mixed product: $(\mathbf{a} \cdot [\mathbf{b} \times \mathbf{c}]) =$

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

(other notation: $(\mathbf{a}, \mathbf{b}, \mathbf{c})$)

properties:

$$(\mathbf{a} \cdot [\mathbf{b} \times \mathbf{c}]) = (\mathbf{b} \cdot [\mathbf{c} \times \mathbf{a}]) = (\mathbf{c} \cdot [\mathbf{a} \times \mathbf{b}]) = -(\mathbf{a} \cdot [\mathbf{c} \times \mathbf{b}]) = -(\mathbf{b} \cdot [\mathbf{a} \times \mathbf{c}]) = -(\mathbf{c} \cdot [\mathbf{b} \times \mathbf{a}])$$

Triple vector product:

$$[\mathbf{a} \times [\mathbf{b} \times \mathbf{c}]] = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

B. Tensor notations:

$\mathbf{a} \Rightarrow a_i; \sum a_i b_i \Rightarrow a_i b_i$ (repetition of a index implies summation)

$$\begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix} \Rightarrow a_i b_j$$

Special symbols:

Kronecker symbol:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Antisymmetric 3-d rank pseudo tensor:

$$e_{ijk} = \begin{cases} 1 & \text{if } (ijk) = (xyz), (zxy), (yzx) \text{ - cyclic order} \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \\ -1 & \text{if } (ijk) = (yxz), (xzy), (zyx) \text{ anticyclic order} \end{cases}$$

Properties:

D. Special coordinates

$$\begin{aligned} \delta_{ij} &= \delta_{ji}; \delta_{ii} = 3; \\ \delta_{ij} A_{iklm\dots} &= A_{jklm\dots} \\ e_{\alpha\beta\gamma} &= -e_{\beta\alpha\gamma} = e_{\beta\gamma\alpha}; e_{\alpha\alpha\beta} = 0; \\ e_{\alpha\beta\gamma} e_{\mu\nu\gamma} &\equiv e_{\alpha\beta\gamma}^{\mu\nu\gamma} = \delta_{\alpha\mu} \delta_{\beta\nu} - \delta_{\alpha\nu} \delta_{\beta\mu}; \\ \delta_{ij} e_{ijk} &= 0; \end{aligned}$$

Product of vectors:

$$\begin{aligned} (\mathbf{a} \cdot \mathbf{b}) &= a_i b_i = \delta_{ij} a_i b_j \\ [\mathbf{a} \times \mathbf{b}]_i &= e_{ijk} a_j b_k \\ (\mathbf{a} \cdot [\mathbf{b} \times \mathbf{c}]) &= e_{ijk} a_i b_j c_k \end{aligned}$$

C. Vector analysis:

Differential operators:

$$\begin{aligned} \vec{\nabla} &= \partial_i = (\partial_x, \partial_y, \partial_z), \\ \text{grad } \varphi &\equiv \nabla \varphi \quad (\text{grad } \varphi)_i = \partial_i \varphi \\ \text{div } \vec{a} &\equiv (\nabla \vec{a}) = \partial_i a_i, \\ \text{rot } \vec{a} &\equiv [\nabla, \vec{a}]; \quad (\text{rot } \vec{a})_\alpha = e_{\alpha\beta\gamma} \partial_\beta a_\gamma \end{aligned}$$

Laplacian:

$$\nabla \cdot \nabla = \Delta = \partial_i \partial_i = \partial_x^2 + \partial_y^2 + \partial_z^2$$

Properties:

$$\nabla r = \frac{\vec{r}}{r} = \vec{n}; \quad \nabla f(r) = f'(r) \cdot \vec{n};$$

$$\text{div } \vec{r} = 3; \quad \text{rot } \vec{r} = 0$$

$$(\vec{a} \nabla) \vec{r} = \vec{a}; \quad \nabla(\vec{a} \vec{r}) = \vec{a}; \quad \text{rot}[\vec{a} \vec{r}] = 2 \vec{a}; \quad (\vec{a} = \text{const})$$

where: $\vec{r} = (x^2 + y^2 + z^2)^{1/2}$ - radius vector, $r = |\vec{r}|$.

Example of calculations:

$$\begin{aligned} \Delta \frac{1}{r^2} &= \partial_i \partial_i \frac{1}{r^2} = -2 \partial_i \frac{r_i}{r^4} = -2 \frac{1}{r^4} \partial_i r_i - 2 r_i \partial_i \frac{1}{r^4} = \\ &= -\frac{6}{r^4} + 8 r_i \frac{r_i}{r^6} = \frac{2}{r^4} \end{aligned}$$

Gauss Theorem:

$$\iiint \text{div } \vec{a} dV = \oint (\vec{a} d\vec{s})$$

Stocks Theorem:

$$\oint (\vec{a} d\vec{l}) = \iiint_S (\text{rot } \vec{a} d\vec{s})$$

Spherical coordinates: r, θ, φ ; ($r = \sqrt{x^2 + y^2 + z^2}$)

$$\begin{aligned} z &= r \cos \theta \\ x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \end{aligned}$$

Operators:

$$\text{grad } \Phi = (\partial_r \Phi) \mathbf{i}_r + \frac{1}{r} (\partial_\theta \Phi) \mathbf{i}_\theta + \frac{1}{r \sin \theta} (\partial_\varphi \Phi) \mathbf{i}_\varphi$$

$$\text{div } \mathbf{F} = \frac{1}{r^2} \partial_r (r^2 F_r) + \frac{1}{r \sin \theta} \partial_\theta (F_\theta \sin \theta) + \frac{1}{r \sin \theta} \partial_\varphi F_\varphi$$

$$\begin{aligned} \text{rot } \mathbf{F} &= \frac{1}{r \sin \theta} [\partial_\theta (F_\varphi \sin \theta - \partial_\varphi F_\theta)] \mathbf{i}_r \\ &+ \frac{1}{r} \left[\frac{1}{\sin \theta} \partial_\varphi F_r - \partial_r (r F_\varphi) \right] \mathbf{i}_\theta + \frac{1}{r} [\partial_r (r F_\theta) - \partial_\theta F_r] \mathbf{i}_\varphi \end{aligned}$$

$$\Delta \Phi = \frac{1}{r^2} \partial_r (r^2 \partial_r \Phi) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\theta \Phi) + \frac{1}{r^2 \sin^2 \theta} \partial_\varphi^2 \Phi$$

Cylindrical coordinates: z, ρ, φ ; ($\rho = \sqrt{x^2 + y^2}$)

$$\begin{aligned} z &= z \\ x &= \rho \cos \varphi \\ y &= \rho \sin \varphi \end{aligned}$$

Operators:

$$\text{grad } \Phi = (\partial_\rho \Phi) \mathbf{i}_\rho + \frac{1}{\rho} (\partial_\varphi \Phi) \mathbf{i}_\varphi + (\partial_z \Phi) \mathbf{i}_z$$

$$\text{div } \mathbf{F} = \frac{1}{\rho} \partial_\rho (\rho F_\rho) + \frac{1}{\rho} \partial_\varphi F_\varphi + \partial_z F_z$$

$$\begin{aligned} \text{rot } \mathbf{F} &= \left[\frac{1}{\rho} \partial_\varphi F_z - \partial_z F_\varphi \right] \mathbf{i}_\rho \\ &+ [\partial_z F_\rho - \partial_\rho F_z] \mathbf{i}_\varphi + \frac{1}{\rho} [\partial_\rho (\rho F_\varphi) - \partial_\varphi F_\rho] \mathbf{i}_z \end{aligned}$$

$$\Delta \Phi = \frac{1}{\rho} \partial_\rho (\rho \partial_\rho \Phi) + \frac{1}{\rho^2} \partial_\varphi^2 \Phi + \partial_z^2 \Phi$$

E. Problems:

1. Simplify, using both the vector algebra and tensor notation technique:

$[\mathbf{a} \times \mathbf{b}] \cdot [\mathbf{c} \times \mathbf{d}]$; consider also the particular case: $[\mathbf{a} \times \mathbf{b}]^2$

2. Calculate:

a) $e_{\alpha\beta\gamma} e_{\mu\beta\gamma}$

b) $e_{\alpha\beta\gamma} e_{\alpha\beta\gamma}$

3. Prove the following formulas, using both the vector algebra and tensor notation technique:

a) $\text{rot rot } \vec{a} \equiv [\nabla, [\nabla, \vec{a}]] = \nabla(\nabla\vec{a}) - (\nabla\nabla)\vec{a} \equiv \text{grad div } \vec{a} - \Delta\vec{a}$

b) $\text{div} [\vec{a}\vec{b}] \equiv (\nabla, [\vec{a}\vec{b}]) = (\vec{b}[\nabla\vec{a}]) - (\vec{a}[\nabla\vec{b}]) \equiv \vec{b} \text{rot } \vec{a} - \vec{a} \text{rot } \vec{b}$,

c) $\text{rot} [\vec{a}\vec{b}] \equiv [\nabla, [\vec{a}\vec{b}]] = (\vec{a}(\nabla\vec{b})) - (\vec{b}(\nabla\vec{a})) - (\vec{a}\nabla)\vec{b} + (\vec{b}\nabla)\vec{a}$

$= \vec{a} \text{div } \vec{b} - \vec{b} \text{div } \vec{a} - (\vec{a}\nabla)\vec{b} + (\vec{b}\nabla)\vec{a}$

d) $\text{grad} (\vec{a}\vec{b}) \equiv \nabla(\vec{a}\vec{b}) = [\vec{b}[\nabla\vec{a}]] + [\vec{a}[\nabla\vec{b}]] + (\vec{a}\nabla)\vec{b} + (\vec{b}\nabla)\vec{a}$

$= [\vec{a} \text{rot } \vec{b}] + [\vec{b} \text{rot } \vec{a}] + (\vec{a}\nabla)\vec{b} + (\vec{b}\nabla)\vec{a}$

e) $\text{rot } f \vec{a} \equiv [\nabla, f \vec{a}] = [\text{grad } f, \vec{a}] + f \text{rot } \vec{a}$,

f) $\text{div } f \vec{a} \equiv (\nabla, f \vec{a}) = (\vec{a}, \text{grad } f) + f \text{div } \vec{a}$

4. Calculate:

a) $\text{grad}(\sin(\mathbf{kr}))$

b) $\text{div} \frac{\mathbf{r}}{r^n}$

c) $\text{rot} \frac{\mathbf{a} \times \mathbf{r}}{r^m}$

d)* $\text{grad}[(\mathbf{a} \times \mathbf{r}) \cdot (\mathbf{b} \times \mathbf{r})]$

e) $\Delta \frac{1}{r}$

f)* $\Delta \frac{1}{r} e^{-\lambda r^2}$

Answer:

$$\text{grad}[(\mathbf{a} \times \mathbf{r}) \cdot (\mathbf{b} \times \mathbf{r})] = 2\mathbf{r}(\mathbf{ab}) - \mathbf{a}(\mathbf{br}) - \mathbf{b}(\mathbf{ar})$$

$$\Delta \frac{1}{r} e^{-\lambda r^2} = 4\pi\delta(r) + \frac{\lambda}{r} e^{-\lambda r^2} (-2 + 4\lambda r^2)$$

II. THEORY OF RELATIVITY

A. 4-vectors

contravariant 4-vector: $\underline{A}^i \equiv (A^0, A^\alpha) = (A^0, \mathbf{A})$

covariant 4-vector: $\underline{A}_i \equiv (A_0, A_\alpha) \equiv (A^0, -\mathbf{A})$.

Differential operators:

$$\frac{\partial}{\partial x_i} \equiv \nabla^i; \quad \frac{\partial}{\partial x^i} \equiv \nabla_i; \quad \nabla^i \nabla_i \equiv -\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta;$$

Scalar production: $\underline{AB} = A^i B_i = A_0 B_0 - \mathbf{AB}$;

Cinematics of the relativistic particle:

4-radius vector $\underline{x}^i = (ct, \mathbf{r})$

interval: $s = \sqrt{x^i x_i}$; $s^2 = (ct)^2 - (\mathbf{r})^2$; $ds = c dt \sqrt{1 - (v/c)^2}$;

B. Lorentz Transformations: ($\mathbf{v} \parallel x$):

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \beta = \frac{v}{c}$$

Vector of 4-velocity:

$$u^i \equiv \frac{dx^i}{ds} = \left(\gamma, \gamma \frac{\mathbf{v}}{c} \right) \quad u^i u_i = 1;$$

m - mass, ε - energy, \mathbf{p} - momentum

$$\varepsilon = mc^2 \gamma, \quad \mathbf{p} = m\mathbf{v}\gamma, \quad \mathbf{v} = \frac{c^2 \mathbf{p}}{\varepsilon}, \quad d\varepsilon = \mathbf{v} d\mathbf{p}.$$

energy in the rest:

$$\varepsilon = mc^2;$$

Relativistic invariant:

$$\underline{p}^2 = p^i p_i = (\varepsilon/c)^2 - (\mathbf{p})^2 = m^2 c^2.$$

C. Relativistic formalism in the field theory

1. Equations of the electromagnetic field:

4 - potential, $A^i = (\varphi, \mathbf{A})$, φ - scalar potential \mathbf{A} - vector potential

$$\text{Electric field: } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi,$$

$$\text{Magnetic field } \mathbf{H} = \text{rot } \mathbf{A}.$$

Gauge transformations:

$$\mathbf{A}' = \mathbf{A} + \nabla f;$$

$$\varphi' = \varphi - \partial f / c \partial t$$

Tensor of the Electromagnetic Field :

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

$$F_{ik} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -H_z & H_y \\ -E_y & H_z & 0 & -H_x \\ -E_z & -H_y & H_x & 0 \end{pmatrix};$$

$$F_{ik}(\mathbf{E}, \mathbf{H}) = F^{ik}(-\mathbf{E}, \mathbf{H})$$

Dual Tensor:

$$\tilde{F}_{ik} = \frac{1}{2} e_{iklm} F^{lm}$$

$$\tilde{F}_{ik}(\mathbf{E}, \mathbf{H}) \Rightarrow F^{ik}(\mathbf{E} \rightarrow -\mathbf{H}, \mathbf{H} \rightarrow -\mathbf{E})$$

$e^{0123} = -e_{0123} = 1$, e^{ijkl} - antisymmetric 4-rank tensor.

Relativistic Invariants of the Electromagnetic field:

Scalar:

$$F^{ik} F_{ik} = 2(\mathbf{H}^2 - \mathbf{E}^2)$$

Pseudo-Scalar:

$$F^{ik} \tilde{F}_{ik} = -4(\mathbf{H}\mathbf{E})$$

Maxwell's Equations:

$$\begin{aligned} \frac{\partial F^{ik}}{\partial x^k} &= -\frac{4\pi}{c} j^i & \text{div} \mathbf{E} &= 4\pi\rho, & \text{rot} \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j} \\ \frac{\partial \tilde{F}^{ik}}{\partial x^k} &= 0 & \text{div} \mathbf{H} &= 0, & \text{rot} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \end{aligned}$$

4-current: $j^i = (c\rho, \mathbf{j})$

Energy-Momentum tensor of the electromagnetic field.

$$\begin{aligned} 0 &= \frac{1}{c} \frac{\partial}{\partial t} \frac{[\mathbf{E}\mathbf{H}]_\alpha}{4\pi} + \frac{\partial}{\partial x_\beta} \sigma_{\alpha\beta} \\ \sigma_{\alpha\beta} &= -W \delta_{\alpha\beta} + \frac{E_\alpha E_\beta + H_\alpha H_\beta}{4\pi}. \end{aligned}$$

2. Motion of the relativistic charged particle in the Electromagnetic field

Motion Equation:

$$\text{Covariant form: } mc \frac{du^i}{ds} = \frac{e}{c} F^{ik} u_k$$

3-dimentional form:

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{v}\mathbf{H}]; \quad \frac{d\varepsilon}{dt} = e\mathbf{E}\mathbf{v}$$

A. Equations of the electromagnetic field

Maxwell's Equations:

$$\begin{aligned} \text{div} \mathbf{E} &= 4\pi\rho, & \text{rot} \mathbf{H} &= \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} + \frac{4\pi}{c} \mathbf{j} \\ \text{div} \mathbf{H} &= 0, & \text{rot} \mathbf{E} &= -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{H} \end{aligned}$$

Density W and flux \mathbf{S} of the electromagnetic field energy:

$$\frac{\partial W}{\partial t} + \text{div} \mathbf{S} = 0; \quad W = \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi}; \quad \mathbf{S} = \frac{c}{4\pi} [\mathbf{E}\mathbf{H}]$$

B. Electric field of static charges (multipole expansion)

Potential:

$$\varphi = \frac{e}{r} + \frac{d_\alpha n_\alpha}{r^2} + \frac{d_{\alpha\beta} n_\alpha n_\beta}{2r^3} + \dots$$

Electric field:

$$E_\alpha = -\partial_\alpha \varphi = \frac{en_\alpha}{r^2} + \frac{3(d_\beta n_\beta) n_\alpha - d_\alpha}{r^3} + \dots$$

where:

Charge:

$$e = \sum_a e_a$$

Dipolar moment

$$d_\alpha = \sum_a e_a r_\alpha^a$$

Quadrupole moment

$$D_{\alpha\beta} = \sum_a e_a (3r_\alpha^a r_\beta^a - (r^a)^2 \delta_{\alpha\beta});$$

Energy of the dipole in the external field:

$$V_e = -(\mathbf{d}\mathbf{E}).$$

Momentum, acting on the dipole in the external field:

$$\mathbf{K} = [\mathbf{d}\mathbf{E}]$$

Force, acting on the dipole in the external field:

$$\mathbf{F} = (\mathbf{d}\nabla)\mathbf{E}$$

Energy of the dipolar interaction:

$$U = \frac{(\mathbf{d}_1 \mathbf{d}_2) - 3(\mathbf{d}_1 \mathbf{n}_{12})(\mathbf{d}_2 \mathbf{n}_{12})}{r^3}$$

Biot-Savart law:

$$d\mathbf{H} = \frac{1}{c} \frac{[d\mathbf{j} \times \mathbf{n}]}{r^2}$$

Magnetic moment of the closed current loop:

$$\mathbf{m} = \frac{e}{2c} [\mathbf{r} \times \mathbf{j}]$$

$\mathbf{m} = g\mathbf{L}$; (\mathbf{L} - orbital momentum, g - gyromagnetic ratio).

Magnetic moment of spin 1/2:

$$\mathbf{m} = g_0 \mu_B \mathbf{S}$$

where the g -factor $g = 2$, the Bohr magneton $\mu_B = e\hbar/2mc = 10^{-24} J/T$

Dipolar magnetic field from the magnetic moment:

$$\mathbf{H} = \frac{3(\mathbf{m}\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3}$$

The energy of the magnetic dipole in the external field:

$$V = -(\mathbf{m}\mathbf{H})$$

D. Plane waves:**Wave equations:**

$$\Delta \mathbf{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{E} = 0$$

$$\Delta \mathbf{H} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{H} = 0$$

Solution:

$$\mathbf{E} = \mathbf{E}_0 f(\mathbf{k}\mathbf{r} - \omega t),$$

$$\mathbf{H} = \mathbf{H}_0 f(\mathbf{k}\mathbf{r} - \omega t)$$

$$\mathbf{E} = [\mathbf{H}\mathbf{n}]; \quad \mathbf{n} = \frac{\mathbf{k}}{k}; \quad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}.$$

Fourier expansion over the plane waves

$$\mathbf{E} = \sum_{\mathbf{k}, \alpha} \mathbf{E}_{\mathbf{k}, \alpha} e^{i(\mathbf{k}\mathbf{r} - \omega t)},$$

$$\mathbf{H} = \sum_{\mathbf{k}, \alpha} \mathbf{H}_{\mathbf{k}, \alpha} e^{i(\mathbf{k}\mathbf{r} - \omega t)}$$

where α is the polarization index and

$$\mathbf{E}_{\mathbf{k}, \alpha} = [\mathbf{H}_{\mathbf{k}, \alpha} \mathbf{n}]; \quad \omega = \omega(\mathbf{k}) = ck$$

A. Problems:

1*. By direct calculations:

a) Show that potential of quadrupole:

$$\varphi(r) = D_{ij} \frac{n_i n_j}{2r^3}$$

satisfies the Laplace equation:

$$\Delta \varphi = 0$$

b) Demonstrate that electric field of quadrupole satisfies the Maxwell equation: $\text{rot} \mathbf{E} = 0$

2*. Four positive charges $+e$ located in vertices of rectangular $2a \times 2b$ are compensated by one central negative charge $-4e$.

a) Find potential $\varphi(r)$ of the electric field.

b) Find electric fields $\mathbf{E}(x)$, $\mathbf{E}(y)$, $\mathbf{E}(z)$ along the principal axes x, y, z of rectangular.

Answer:

$$\varphi(r) = 2e \frac{(2a^2 - b^2)x^2 + (2b^2 - a^2)y^2 - (a^2 + b^2)z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\mathbf{E}(x) = 6e \frac{2a^2 - b^2}{x^4} \mathbf{n}_x$$

$$\mathbf{E}(y) = 6e \frac{2b^2 - a^2}{y^4} \mathbf{n}_y$$

$$\mathbf{E}(z) = -6e \frac{a^2 + b^2}{z^4} \mathbf{n}_z$$

3a) Find the potential and electric field from the four point charges, located in the apexes of the square and having alternatively the charges $\pm e$

3b) Find the potential and field from the three charges $+e$ in the apexes of the equilateral triangular, compensated by the charge $-3e$ in its center.

Indication: use the quadrupole approximation.

4. Determine the motion of a charge in constant uniform electric and magnetic fields (3-8).

Solution:

Equation of motion:

$$m \frac{d\mathbf{v}}{dt} = e\mathbf{E} + \frac{e}{c} [\mathbf{v}\mathbf{H}]$$

Particular cases:

a) $H = 0$: linear motion

$$x = eEt^2/2m$$

b) $E = 0$: circular motion with:

$$R = \frac{cmv}{eH}; \quad \omega_c = \frac{eH}{mc}$$

where ω_c is the cyclotron frequency

General case, $\mathbf{H} \parallel \mathbf{z}$, \mathbf{E} is in the $\mathbf{y} - \mathbf{z}$ plane.

$$\begin{aligned} m \ddot{x} &= \frac{e}{c} \dot{y} H \\ m \ddot{y} &= eE_y - \frac{e}{c} \dot{x} H \\ m \ddot{z} &= eE_z \end{aligned}$$

The last equation gives: $z = eE_z t^2/2m$

From the first two equations we have:

$$\frac{d}{dt}(\dot{x} + i\dot{y}) + i\omega_c(\dot{x} + i\dot{y}) = i\frac{e}{m}E_y$$

Solution:

$$\dot{x} + i\dot{y} = ae^{i(\omega_c t + \varphi)} + c\frac{E_y}{H}$$

Then,

$$\begin{aligned} \dot{x} &= a \cos(\omega_c t + \varphi) + c\frac{E_y}{H} \\ \dot{y} &= -a \sin(\omega_c t + \varphi) \end{aligned}$$

Initial conditions: $v_y(0) = 0$ and $v_x(0) = v_0$ gives $\varphi = 0$, and $a = v_0 - c\frac{E_y}{H}$.

Taking $x(0) = y(0) = 0$, we obtain:

$$\begin{aligned} x &= \frac{a}{\omega_c} \sin(\omega_c t + \varphi) + c\frac{E_y}{H}t \\ y &= \frac{a}{\omega_c} (\cos(\omega_c t + \varphi) - 1) \end{aligned}$$

Generally, the motion is along the *trochoid*; in particular case: $v_0 = 0$, the motion is along the *cycloid*

Drift velocity:

$$\mathbf{V} = c\frac{[\mathbf{E} \times \mathbf{H}]}{H^2}$$

Condition of application:

$$E_y \ll H$$

5a. Find the induced dipolar moment of the Hydrogen atom in the electric field:

Answer:

The induced dipolar moment of atom in the electric field:

$$\mathbf{d} = \alpha \mathbf{E}$$

is defined by the electric polarizability

$$\alpha \simeq a_B^3$$

where the Bohr radius $a_B = \hbar^2/me^2 = 0.5\text{\AA}$

5b. Estimate the induced magnetic moment of the Hydrogen atom in the magnetic field:

Solution:

Orbital precession of the electron in the external field with the cyclotron frequency:

$$\omega_c = \frac{eH}{mc}$$

produces the circular current:

$$I = -e\frac{\omega_c}{2\pi} = -\frac{e^2 H}{2\pi mc}$$

and hence the magnetic moment:

$$m = \pi \rho^2 I / c = -\frac{e^2 H \rho^2}{2mc^2}$$

ρ is roughly estimated as the Bohr radius: $a_B = \hbar^2/me^2 = 0.5\text{\AA}$, then, the polarization

$$\beta \simeq \left(\frac{e^2}{\hbar c^2}\right)^2 a_B^3$$

where the fine structure constant $e^2/\hbar c^2 = 1/137$

V. RADIATION OF THE ELECTROMAGNETIC WAVES.

A. Plane waves:

$$\mathbf{H} = \frac{1}{c}[\dot{\mathbf{A}} \times \mathbf{n}], \quad \mathbf{E} = \frac{1}{c}[[\dot{\mathbf{A}} \times \mathbf{n}] \times \mathbf{n}]$$

$$\mathbf{E} = [\mathbf{H} \times \mathbf{n}]; \quad \mathbf{n} = \frac{\mathbf{k}}{k}; \quad k = \frac{\omega}{c} = \frac{2\pi}{\lambda}.$$

B. Retarded potentials:

$$\varphi(\vec{r}, t) = \int \frac{\rho(\vec{r}', t - R/c)}{R} d^3 r';$$

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \frac{\vec{j}(\vec{r}', t - R/c)}{R} d^3 r';$$

Where $R = |\vec{r} - \vec{r}'|$

At the distances larger then the system size and then the wave length (*wave zone*), the electromagnetic field is approximated by plane waves.

Applicability: The size of the system should be smaller than radiated wavelength, the distance to the system should be larger than the size of the system.

The fields in the wave zone:

$$\mathbf{A} = \frac{1}{cR_0} \dot{\mathbf{d}}$$

$$\mathbf{H} = \frac{1}{c^2 R_0} [\ddot{\mathbf{d}} \times \mathbf{n}], \quad \mathbf{E} = \frac{1}{c^2 R_0} [[\ddot{\mathbf{d}} \times \mathbf{n}] \times \mathbf{n}]$$

Intensity of the **dipolar radiation into the solid angle** $d\varpi$:

$$dI = \frac{1}{4\pi c^3} [\ddot{\mathbf{d}} \times \mathbf{n}]^2 d\varpi$$

The total radiation intensity:

$$I = \frac{2}{3c^3} \ddot{\mathbf{d}}^2;$$

Radiation of charge moving with acceleration w :

$$I = \frac{2e^2 w^2}{3c^3}$$

D. Radiation damping:

Applicability: Dipolar radiation; damping force should be smaller than the acting force, the motion should be finite.

Damping force:

$$\mathbf{f} = \frac{2e}{3c^3} \ddot{\mathbf{d}}$$

Example: Equation of motion of charge in the electromagnetic field:

$$m \dot{\mathbf{v}} = e\mathbf{E} + \frac{e}{c} [\mathbf{v} \times \mathbf{H}] + \frac{2e^2}{3c^3} \ddot{\mathbf{v}}$$

E. Scattering by free charges:

The scattering is characterized by the ratio of the amount of energy emitted in a given direction per unit time, to the energy flux density of the incident radiation, that is by the **differential cross section** $d\sigma$ of polarized wave into the solid angle $d\varpi$:

$$d\sigma = \frac{\overline{dI}}{S} = \left(\frac{e^2}{mc^2} \right)^2 [\mathbf{e} \times \mathbf{n}]^2 d\varpi$$

Where $\mathbf{e} = \mathbf{E}/E$, $\mathbf{n} = \mathbf{r}/r$.

The total cross section:

$$\sigma = \int d\sigma = \frac{8\pi}{3} \left(\frac{e^2}{mc^2} \right)^2$$

1. Charge $(-e, m)$ is rotated around the charge $+e$ by a circular orbit of radius r . Find the radiation energy loss during one period. How the distance between charges depends on the time? Determine the life-time of the system. (9-9)

Solution:

Equation of motion (without radiation):

$$\frac{e^2}{r^2} = m\ddot{r} = m \frac{v^2}{r}$$

Then,

$$r = e^2/mv^2$$

Energy:

$$E = \frac{mv^2}{2} - \frac{e^2}{r} = -\frac{e^2}{2r}$$

Energy loss per unit time:

$$\begin{aligned} I &= -\frac{dE}{dt} = \frac{2}{3c^3} \ddot{\mathbf{d}}^2 = \frac{2e^2}{3c^3} \ddot{r}^2 \\ &= \frac{2e^2}{3c^3} \left(\frac{e^2}{mr^2} \right)^2 = \mu E^2 \end{aligned}$$

where

$$\mu = \frac{32}{3c^3 m^2 e^2}$$

Energy loss per one rotation:

$$\Delta E = -IT = -I \frac{2\pi}{\omega} = -2\pi I \frac{r}{v} = -\frac{4\pi m v^5}{3c^3}$$

$$\frac{\Delta E}{E} = \frac{8\pi}{3} \left(\frac{v}{c} \right)^3 \sim \left(\frac{e}{\hbar c} \right)^2 = \alpha^3 \sim 10^{-6}$$

Solving Diff. Eq.

$$-\frac{dE}{dt} = \mu E^2$$

We obtain:

$$E(t) = E_0 \left(\frac{t^* - t}{t^*} \right)^{-1/3}$$

With time of life

$$t^* = -\frac{1}{3\mu E_0^3} \sim 10^{-10} \text{ s}$$

Then,

$$r = r_0 \left(\frac{t^* - t}{t^*} \right)^{1/3}$$

2*. The charged perfect harmonic oscillator (m, ω_0) irradiates the electromagnetic waves. Find :

- The distribution of electric field $\mathbf{E}(r, t)$ in a space
- The distribution of magnetic field $\mathbf{H}(r, t)$ in a space
- The energy, $\Delta E/E$ irradiated by the charged perfect harmonic during one oscillation.
- How the amplitude A of oscillations decreases with time, because of the irradiation effects?

Answer:

$$\frac{\Delta E}{E} = -\frac{4\pi}{3} \frac{e^2}{mc^3} \omega_0$$

$$A(t) = A_0 e^{-t/\tau}; \quad \tau = \frac{3}{4} \frac{mc^3}{e^2 \omega_0^2}$$

3*. Charge (e, m) is moving in magnetic field \mathbf{H} with initial velocity \mathbf{v}_0 find:

- Trajectory without radiation.
- Total radiation per unit time,
- Angular distribution of radiation
- Time dependence of the charge energy and of the radius vector. Whether the charge will stop?
- How the answer will modify if use the quantum mechanics?

Answer:

$$I = \frac{2e^2}{3c^3} \omega^4 r^2; \quad \omega = \frac{e}{mc} H$$

$$dI = \frac{e^2 \omega^2 r^2}{2} (1 + \cos^2 \theta) \sin \theta d\theta$$

$$r(t) = r_0 \exp\left(-\frac{4e^2}{3c^3 m} \omega^2 t\right)$$

In classical electrodynamics, the charge never stop; in quantum mechanics the stopping time:

$$t_c = \frac{3c^3 m}{4e^2 \omega^2} \ln \frac{r_0}{r^*}$$

where the cut-off time is the cyclotron radius of the 1-th Landau level:

$$r^* = \sqrt{\hbar/m\omega}$$

4. Determine the effective cross section for scattering of a linearly polarized wave by an oscillator with damping, (9-11)

Solution:

Equation of motion:

$$\ddot{r} + \gamma \dot{r} + \omega_0^2 r = \frac{e}{m} E_0 e^{-i\omega t}$$

Solution:

$$r = \frac{e}{m} E_0 \frac{e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

Radiation intensity:

$$I = \frac{2}{3c^3} \dot{\mathbf{d}}^2 = \frac{2e^2}{3c^3} \dot{r}^2$$

The cross section:

$$\sigma = \frac{I}{S} = \frac{8\pi}{3} \left(\frac{e}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}$$

VI. ELECTROSTATICS

A. Parameters

microscopic:

- microscopic electric fields, $\mathbf{e}(r)$
- microscopic (free, unbound) charges q_i
- microscopic dipoles \mathbf{d}_i

macroscopic:(averaged over the molecular scale)

- average field $\mathbf{E} = \langle \mathbf{e} \rangle$,
- density of the free charges: $\rho = \langle q_i \rangle$
- dipolar moment (per unit volume): $\mathbf{P} = \langle \mathbf{d}_i \rangle$

global:

- External field (without the body), \mathbf{E}_0
- Dipolar momentum of the body, $\mathbf{d} = \sum \mathbf{d}_i = \int \mathbf{P} dV$

B. Electrostatic equations in metal:

$$\text{div} \mathbf{E} = 0, \quad \text{rot} \mathbf{E} = 0$$

for potential φ :

$$\mathbf{E} = -\text{grad} \varphi, \quad \Delta \varphi = 0$$

Inside the metal, $\mathbf{E} = 0$; all the charges are located on the surface.

The boundary condition on the surface:

$$\mathbf{E}_t = 0, \quad \mathbf{E}_n^- = 0, \quad \mathbf{E}_n^+ = 4\pi\sigma$$

C. Electrostatic equations in dielectric:

The **dielectric polarization** \mathbf{P} is produced by distribution of internal charges $\bar{\rho}$ inside of dielectric and is defined as: $\text{div}\mathbf{P} = -\bar{\rho}$

The **electric induction**: $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$

The averaging of the microscopic Maxwell's equations gives:

$$\text{div}\mathbf{D} = \rho_{ex}, \quad \text{rot}\mathbf{E} = 0$$

ρ_{ex} - external charge introduced to dielectric.

The **boundary conditions on the surface**:

$$\Delta\mathbf{E}_t = 0, \quad \Delta\mathbf{D}_n = 0$$

D. The dielectric constant ε

1. Simplest approximation:

In the linear approximation, the induction and field are related by the **dielectric constant** (=dielectric permittivity) ε . Generally $\varepsilon > 0$, the metal can be formally considered as dielectric with $\varepsilon \rightarrow \infty$.

For the isotropic media ε is a scalar:

$$\mathbf{D} = \varepsilon\mathbf{E}$$

Then, \mathbf{P} is also proportional to the field:

$$\mathbf{P} = \kappa\mathbf{E}$$

with **dielectric susceptibility** (=polarization coefficient)

$$\kappa = (\varepsilon - 1)/4\pi$$

2. Charge distribution in the nonuniform media

$$\rho = -\text{div}\mathbf{P} = -\text{div}\frac{\varepsilon - 1}{4\pi\varepsilon}\mathbf{D} = -\frac{\mathbf{E}}{4\pi\varepsilon}\nabla\varepsilon$$

Inside of the uniform media $\varepsilon = 0$, all the charges are on the surface.

3. More realistic approximations (examples):

- anisotropy

For the crystals, dielectric constant and susceptibility are 2-nd rank tensors:

$$D_i = \varepsilon_{ij}E_j, \quad D_i = \kappa_{ij}E_j$$

- nonlinearity

$$\mathbf{D} = \varepsilon\mathbf{E} + \varepsilon_3 E^2\mathbf{E} + \dots$$

- time dispersion

$$\mathbf{D}(\mathbf{r}, t) = \int_0^\infty \varepsilon(t')\mathbf{E}(\mathbf{r}, t - t')dt'$$

E. Electrostatic potential

Equation for φ :

$$\nabla(\varepsilon\nabla\varphi) = 0$$

when the media is uniform,

$$\Delta\varphi = 0$$

Boundary conditions:

$$\varphi_1 = \varphi_2, \quad \varepsilon_1 \frac{\partial\varphi_1}{\partial n} = \varepsilon_2 \frac{\partial\varphi_2}{\partial n}$$

F. Electric field inside the body.

- The electric field \mathbf{E} inside of dielectric can be presented as composition of the (uniform) external applied, field \mathbf{E}_0 and (nonuniform) **depolarization field** \mathbf{E}_1 that, generally, is directed against \mathbf{E}_0 .

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$$

\mathbf{E}_1 and, hence, \mathbf{E} depend on the **geometry** of the body. In the important case of ellipsoid, $\mathbf{E} = (E_x, E_y, E_z)$ is uniform and equal to:

$$E_i = \gamma_{ij}E_{0,j}$$

If the x, y, z axes are taken along principal axes a, b, c of ellipsoid, γ_{ij} is the diagonal tensor with

$$\gamma_{ii} = \frac{4\pi}{4\pi + (\varepsilon - 1)N_i}$$

where N_x, N_y, N_z ($N_x + N_y + N_z = 4\pi$) are the **depolarization factors** that depend only on geometrical parameters a, b, c .

In some particular cases (Problem 1):

for the plate and cylinder that are parallel to the field

$$\mathbf{E} = \mathbf{E}_0$$

for the plate that is perpendicular to the field

$$\mathbf{E} = \mathbf{E}_0/\varepsilon$$

for the sphere

$$\mathbf{E} = 3\mathbf{E}_0/(\varepsilon + 2)$$

G. Energy of the dielectric in the field:

$$F = \frac{1}{8\pi} \int \mathbf{E}\mathbf{D}dV = \frac{\varepsilon}{8\pi} \int \mathbf{E}^2 dV$$

"the full free energy" \mathcal{F} of the system:

$$\mathcal{F} = \int (F - \mathbf{E}_0^2/8\pi)dV$$

the energy of the body with dipolar momentum \mathbf{d} in the external field \mathbf{E}_0 :

$$\begin{aligned} F &= \frac{1}{8\pi} \int (\mathbf{E}\mathbf{D} - \mathbf{E}_0^2)dV \\ &= \frac{1}{8\pi} \int \mathbf{E}_0(\mathbf{D} - \mathbf{E}_0)dV + \frac{1}{8\pi} \int \mathbf{E}(\mathbf{D} - \mathbf{E}_0)dV \\ &\quad - \frac{1}{8\pi} \int \mathbf{E}_0(\mathbf{D} - \mathbf{E})dV \\ &= -\frac{1}{2}\mathbf{d}\mathbf{E}_0 \end{aligned}$$

H. The local fields:

The **local** field, \mathbf{e} , is not equal to the average field \mathbf{E} .

$$\mathbf{e}_{loc} = \mathbf{E} + \mathbf{E}_{lor} + \mathbf{E}_{near}$$

Where the Lorenz field \mathbf{E}_{lor} is the field of charges located on the surface of fictive spherical shell of radius R around the site and \mathbf{E}_{near} is field of the dipolar moments inside the shell.

$$\begin{aligned} \mathbf{E}_{lor} &= \frac{1}{R^2} \int \sigma dS = \int \mathbf{n}(\mathbf{P}\mathbf{n})dS \\ &= \frac{4\pi}{3}\mathbf{P} = \frac{\varepsilon - 1}{3}\mathbf{E} \end{aligned}$$

and

$$\mathbf{E}_{near} = \sum_{r_i < R} (3(\mathbf{p}_j\mathbf{n}_j)\mathbf{n}_i - \mathbf{p}_i)/r_i^3$$

\mathbf{E}_{near} depends on the geometry of the crystal lattice and on the position of the site. $\mathbf{E}_{near} = 0$ for the central site of the cubic lattice or for the random distribution of dipoles in liquid. Then,

$$\mathbf{e}_{loc} = \frac{\varepsilon + 2}{3}\mathbf{E},$$

that is different from the macroscopic field \mathbf{E} .

I. Polarizability

The dipolar moment of the molecule (atom) inside the body is defined by the **polarizability** coefficient α as:

$$\mathbf{p}_i = \alpha_i \mathbf{e}_{loc}(i)$$

Then, the macroscopic polarization of the media consisted from the equivalent dipoles with density n is:

$$\mathbf{P} = \frac{1}{V} \sum_i \mathbf{p}_i = n\alpha \mathbf{e}_{loc} = n\alpha \frac{\varepsilon + 2}{3}\mathbf{E}$$

Since $\mathbf{P} = \mathbf{E}(\varepsilon - 1)/4\pi$, the *macroscopic* dielectric constant ε and *microscopic* polarizability α are related by **Clausius-Mossotti relation**:

$$\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{4\pi}{3}n\alpha$$

The formula is valid only if $\mathbf{E}_{near} = 0$, like in cubic lattice or in liquid of dipoles.

The macroscopic susceptibility, $\chi = (\varepsilon - 1)/4\pi$ diverges as

$$\chi = \frac{n\alpha}{1 - \frac{4\pi}{3}n\alpha}$$

that leads to the *ferroelectric* ground state, when the dipole concentration, $n > n_c = 3/4\pi\alpha$ ($4\pi/3$ *catastrophe*).

For another dipolar arrangement, the *antiferroelectric* ground state can occur.

J. Microscopic sources of polarizability

-Electron polarizability (Like in Hydrogene atom)

From $\ddot{r} + \omega_0^2 r = (e/m)E$ we find $\alpha = er_0/E = e^2/m\omega_0^2$ were $\omega_0 \sim 10^{15}Hz$ (Ultraviolet)

It is also estimated as: $\alpha \sim a_B^3 \sim 10^{-24}cm^3$

-Ion (molecular) polarizability

Similar to the electron polarizability, with $\omega_0 \sim 10^{13}Hz$ (Infrared)

-Orientation polarizability of the polar molecules with permanent dipolar moments $d_0 \sim ea$

$$\alpha = \frac{d_0^2}{3kT}$$

(Langevin-Debye equation, Curie law). At room temperature $\alpha \sim 10^{-23}cm^{-3}$

1. Calculate the electric fields \mathbf{E}_i and \mathbf{E}_e inside and outside of the dielectric sphere in the external field \mathbf{E}_0 . Find the polarization \mathbf{P} inside the sphere, induced dipolar moment and distribution of the charges on the surface of the sphere.

Solution:

Hypothesis:

$$\mathbf{E}_e = \frac{3(\mathbf{d}\mathbf{n})\mathbf{n} - \mathbf{d}}{r^3} + \mathbf{E}_0 \quad \mathbf{d} = D\mathbf{E}_0$$

$$\mathbf{E}_i = A\mathbf{E}_0$$

\mathbf{E}_i and \mathbf{E}_e satisfy the Maxwell equations. Constant A and D are found from the boundary conditions on the surface:

$$\varepsilon \mathbf{E}_{in} = \mathbf{E}_{en}$$

$$\mathbf{E}_{it} = \mathbf{E}_{et}$$

The normal and tangential components of an arbitrary vector \mathbf{C} on the surface of the sphere are calculated as:

$$\mathbf{C}_n = (\mathbf{n}\mathbf{C})\mathbf{n}$$

$$\mathbf{C}_t = \mathbf{C} - (\mathbf{n}\mathbf{C})\mathbf{n}$$

Then,

$$\mathbf{E}_{in} = A(\mathbf{E}_0\mathbf{n})\mathbf{n}$$

$$\mathbf{E}_{it} = A\mathbf{E}_0 - A(\mathbf{E}_0\mathbf{n})\mathbf{n}$$

and

$$\mathbf{E}_{en} = D \frac{2(\mathbf{E}_0\mathbf{n})\mathbf{n}}{R^3} + (\mathbf{E}_0\mathbf{n})\mathbf{n}$$

$$\mathbf{E}_{et} = D \frac{(\mathbf{E}_0\mathbf{n})\mathbf{n} - \mathbf{E}_0}{R^3} + \mathbf{E}_0 - (\mathbf{E}_0\mathbf{n})\mathbf{n}$$

From the boundary conditions, we have:

$$2 \frac{D}{R^3} + 1 = \varepsilon A$$

$$\frac{D}{R^3} - 1 = -A$$

That gives the solution:

$$A = \frac{3}{\varepsilon + 2} \quad D = \frac{\varepsilon - 1}{\varepsilon + 2} R^3$$

Answer:

$$\mathbf{E}_i = \mathbf{E}_0 \frac{3}{\varepsilon + 2}; \quad \mathbf{D}_i = \mathbf{E}_0 \frac{3\varepsilon}{\varepsilon + 2}; \quad \mathbf{P}_i = \mathbf{E}_0 \frac{1}{4\pi} \frac{\varepsilon - 1}{\varepsilon + 2};$$

$$\mathbf{E}_e = \frac{3(\mathbf{d}\mathbf{n})\mathbf{n} - \mathbf{d}}{r^3}; \quad \mathbf{D}_e = \mathbf{E}_e; \quad \mathbf{P}_e = 0;$$

$$\mathbf{d} = \int \mathbf{P}_i dV = \mathbf{E}_0 \frac{1}{3} \frac{\varepsilon - 1}{\varepsilon + 2} R^3$$

$$\sigma = \mathbf{P}\mathbf{n} = E_0 \frac{1}{4\pi} \frac{\varepsilon - 1}{\varepsilon + 2} \cos \theta$$

The solution is *unique* because it satisfies the Maxwell equations and the boundary conditions.

2. Very long cylinder of diameter d and dielectric constant ε is placed in the electric field \mathbf{E}_0 that is perpendicular to the cylinder axis. Find fields ($\mathbf{E}_i, \mathbf{E}_e$) polarizations ($\mathbf{P}_i, \mathbf{P}_e$) inductions ($\mathbf{D}_i, \mathbf{D}_e$) and potentials (φ_i, φ_e) inside and outside of the cylinder. Find its dipolar moment \mathbf{d} and the distribution of the charge σ over the surface.

Answer:

$$\mathbf{E}_i = \mathbf{E}_0 \frac{2}{\varepsilon + 1}; \quad \mathbf{D}_i = \mathbf{E}_0 \frac{2\varepsilon}{\varepsilon + 1}; \quad \mathbf{P}_i = \mathbf{E}_0 \frac{2}{4\pi} \frac{\varepsilon - 1}{\varepsilon + 1};$$

$$\mathbf{E}_e = \mathbf{E}_0 + \frac{2(\mathbf{d}\mathbf{n})\mathbf{n} - \mathbf{d}}{r^3}; \quad \mathbf{D}_e = \mathbf{E}_e; \quad \mathbf{P}_e = 0;$$

$$\mathbf{d} = \int \mathbf{P}_i dV = \mathbf{E}_0 \frac{\varepsilon - 1}{\varepsilon + 1} R^2$$

$$\sigma = \mathbf{P}\mathbf{n} = E_0 \frac{1}{2\pi} \frac{\varepsilon - 1}{\varepsilon + 1} \cos \theta$$

3.: Make the same calculations for the plate with arbitrary angle to the field

VII. THERMODYNAMICS IN ELECTRIC FIELD.

A. Thermodynamic potentials

The free energy, $U = U(S, P, \mathbf{D})$

$$dU = TdS - PdV + \frac{V}{4\pi} \mathbf{E}d\mathbf{D};$$

Thermodynamic potentials: $\Phi = \Phi(T, P, \mathbf{D})$

$$\Phi = U - TS + PV,$$

Then,

$$d\Phi = -SdT + VdP + \frac{V}{4\pi} \mathbf{E}d\mathbf{D};$$

Thermodynamic variables (e.g.)

$$V = \left(\frac{\partial \Phi}{\partial P} \right)_{T, \mathbf{D}} \quad \mathbf{E} = \frac{4\pi}{V} \left(\frac{\partial \Phi}{\partial \mathbf{P}} \right)_{T, D}$$

Cross-relations: (e.g.)

$$\left(\frac{\partial}{\partial P} \frac{V}{4\pi} E\right)_{T,D} = \left(\frac{\partial V}{\partial D}\right)_{T,P} = \left(\frac{\partial^2 \Phi}{\partial D \partial P}\right)_T$$

Other potentials:

Entalpy:

$$W(S, P, \mathbf{D}) = U + PV,$$

Free energy

$$F(T, V, \mathbf{D}) = U - TS,$$

Differential relations:

$$dW = TdS + VdP + \int (\mathbf{E}d\mathbf{D}) \frac{dV}{4\pi};$$

$$dF = -SdT - PdV + \int (\mathbf{E}d\mathbf{D}) \frac{dV}{4\pi};$$

When $\mathbf{D} = \varepsilon \mathbf{E}$:

$$U = U_0 + \frac{V}{8\pi\varepsilon} \mathbf{D}^2 = U_0 + \frac{\varepsilon}{8\pi} \int \mathbf{E}^2 dV$$

$$\Phi = \Phi_0 + \frac{V}{8\pi\varepsilon} \mathbf{D}^2 = \Phi_0 + \frac{\varepsilon}{8\pi} \int \mathbf{E}^2 dV$$

$$F = F_0 + \frac{V}{8\pi\varepsilon} \mathbf{D}^2 = F_0 + \frac{\varepsilon}{8\pi} \int \mathbf{E}^2 dV$$

$$W = W_0 + \frac{V}{8\pi\varepsilon} \mathbf{D}^2 = W_0 + \frac{\varepsilon}{8\pi} \int \mathbf{E}^2 dV$$

REMARK: For practical purposes it is convenient to use the amplitude of the external applied field, \mathbf{E}_0 as an independent variable and consider the induced dipolar moment of the sample $\mathbf{d} = \int \mathbf{P}dV$ as an function of \mathbf{E}_0 .

In this case, one introduces the new thermodynamic potential - "the full free energy" \mathcal{F} of the system:

$$\mathcal{F} = \int (F - \mathbf{E}_0^2/8\pi) dV$$

For which \mathbf{d} and \mathbf{E}_0 are thermodynamically conjugated variables. Then

$$d\mathcal{F} = -SdT - PdV - \mathbf{d}d\mathbf{E}_0;$$

and

$$F = F_0 - \frac{1}{2} \mathbf{d}\mathbf{E}_0$$

Generally, $d_i = \alpha_{ij} E_{j0}$ where α_{ij} is the **polarizability tensor**.

Formulas for other thermodynamic potentials are transformed in an analogous way.

B. Thermodynamic coefficients:

Specific heat, dielectric susceptibility and compressibility:

$$C_{p,E} = -\frac{T}{V} \left(\frac{\partial S}{\partial T}\right)_{p,E} \quad \varepsilon_{p,T} = \left(\frac{\partial D}{\partial E}\right)_{p,T}$$

$$\zeta_{T,E} = -\frac{1}{V} \left(\frac{\partial V}{\partial P}\right)_{T,E}$$

Thermal expansion, pyroelectric coefficient:

$$\alpha_{p,E} = \frac{1}{V} \left(\frac{\partial V}{\partial T}\right)_{p,E} \quad \gamma_{p,E} = \left(\frac{\partial D}{\partial T}\right)_{p,E}$$

Piezoelectric coefficient (particular isotropic case)

$$d_{T,E} = \left(\frac{\partial D}{\partial P}\right)_{T,E} = -\frac{1}{V} \left(\frac{\partial V}{\partial E}\right)_{T,E}$$

C. Equation of state

$$\mathbf{D} = \varepsilon(T)\mathbf{E}$$

$\varepsilon(T) = \text{const}$ for nonpolar dielectrics
 $\varepsilon(T) = C/(T - T_c)$ for polar dielectrics

VIII. MAGNETISM**A. Constant magnetic field and magnetic permeability**

The **magnetization** \mathbf{M} is produced by distribution of microscopic currents internal $\overline{\rho\mathbf{v}}$ and is defined as:
 $\text{rot}\mathbf{M} = \overline{\rho\mathbf{v}}/c$

The **magnetic induction**: $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$

Attention: Although \mathbf{H} is, by analogy with \mathbf{E} , called *the magnetic field*, the true field is really $\mathbf{B} = \langle \mathbf{h} \rangle$ rather than \mathbf{H} .

The averaging of the microscopic Maxwell's equations gives:

$$\text{div}\mathbf{B} = 0, \quad \text{rot}\mathbf{H} = 0$$

The **boundary conditions on the surface**:

$$\Delta\mathbf{H}_t = 0, \quad \Delta\mathbf{B}_n = 0$$

In the linear approximation, the induction and field are related by the **magnetic permeability** μ .

For the isotropic media μ is a scalar:

$$\mathbf{B} = \mu \mathbf{H}$$

Then, \mathbf{P} is also proportional to the field:

$$\mathbf{M} = \chi \mathbf{H}$$

with **magnetic susceptibility** $\chi = (\mu - 1)/4\pi$

For the crystals, permeability and susceptibility are the 2-nd rank tensors:

$$B_i = \mu_{ij} H_j, \quad M_i = \chi_{ij} H_j$$

Generally: $\chi \sim 10^{-6}$

$\chi > 0$ for paramagnets,

$\chi < 0$ for diamagnets,

$\chi = -\infty$ for superconductors.

Thermodynamic relations in magnetic field

Similar as for electrostatics with substitution: $\mathbf{E} \rightarrow \mathbf{H}$, $\mathbf{D} \rightarrow \mathbf{B}$

B. Microscopic sources of magnetism

Van-Leeven theorem: the system of classical charges in thermal equilibrium can not have the magnetic moment, even in the external magnetic field.

Therefore, the magnetism is an essentially quantum-mechanic phenomena.

Principal sources of magnetism:

1) **Spin of electrons:** magnetic moment of one electron with spin $s = 1/2$:

$$\mu = g \mu_B s$$

where the Bohr magneton $\mu_B = e\hbar/2mc$, and "g-factor" is equal to 2.

2) **Orbital movement of electrons:**

a) Angular momentum (for p , d , electrons with un-closed orbital)

b) Momentum, induced by the external magnetic field

C. Type of magnetism:

Magnetic susceptibility:

$$\chi = M/H$$

Diamagnet have $\chi < 0$

Paramagnets have $\chi > 0$

Usually $|\chi| \sim v/c \ll 1$ because of the quantum-mechanic nature of magnetism. Exception are the superconductors..

Ferromagnets have the spontaneous magnetic moment without field.

Nuclear magnetism is produced by magnetic moments of nuclei, it is of 10^{-3} times smaller than the electrons magnetism.

D. Magnetism of the localized electrons (Dielectrics)

1. Langevin diamagnetism

Origin: orbital magnetic moment of the closed atomic orbitals, induced by the external field.

Classical interpretation: Orbital precession of the electron in the external field with the Larmor frequency:

$$\omega_L = \frac{eH}{2mc}$$

produces the circular current:

$$I = -e \frac{\omega_L}{2\pi} = -\frac{e^2 H}{4\pi mc}$$

and hence the magnetic moment:

$$\mu = \pi \rho^2 I / c = -\frac{e^2 H \rho^2}{4mc^2}$$

The magnetic moment of the unit of volume containing n atoms with Z electrons is: $M = Zn\mu$; ρ is roughly estimated as the Bohr radius: \hbar^2/me^2

Therefore:

$$\chi_L \sim -nZ \frac{e^2}{mc^2} \frac{\hbar^4}{m^2 e^4} = -nZ \frac{\hbar^4}{m^3 c^2 e^2}$$

Variation with temperature: temperature independent

2. Van-Vleck paramagnetism:

Origin: orbital magnetic moment of the molecules induced by the excited states of the electrons with nonzero angular momentum, admixed to the ground state because of the influence of magnetic field.

Variation with temperature: temperature independent

$$\chi_P = \frac{n\mu^2}{3\varepsilon_F}$$

Origin: Polarization of the magnetic moments μ of spins and angular momenta of the atoms (molecules) with unclosed electron shells by the external field that are thermally reoriented without field.

Free magnetic moments in external field:

The energy:

$$U = -\mu\mathbf{H} = -\mu H \cos \theta$$

Magnetisation:

$$\mathbf{M} = n \langle \mu \rangle = n\mu \langle \cos \theta \rangle$$

calculations of $\langle \cos \theta \rangle$:

$$\langle \cos \theta \rangle = \frac{\int \cos \theta e^{-\beta U} d\Omega}{\int e^{-\beta U} d\Omega}$$

where $\beta = 1/kT$, $d\Omega = \sin \theta d\varphi d\theta$.

Integration gives:

$$\begin{aligned} \langle \cos \theta \rangle &= \frac{2\pi \int_0^\pi \cos \theta e^{-\beta\mu H \cos \theta} \sin \theta d\theta}{2\pi \int_0^\pi e^{-\beta\mu H \cos \theta} \sin \theta d\theta} \\ &= \frac{\int_{-1}^1 s e^{sx} ds}{\int_{-1}^1 e^{sx} ds} = \frac{d}{dx} \ln \int_{-1}^1 e^{sx} ds \\ &= \frac{d}{dx} \ln (e^{sx} - e^{-sx}) - \frac{d}{dx} \ln x \\ &= \coth x - \frac{1}{x} \equiv L(x) \end{aligned}$$

where $x = \mu H/kT$, $s = \cos \theta$, $L(x)$ - Langevin function

When $H \gg H_{\text{sat}} = kT/\mu$, $L(x) \rightarrow 1$

When $H \ll H_{\text{sat}} = kT/\mu$, $L(x) \approx x/3$

For $T = 1K$:

$H_{\text{sat}} = 1.6T$ for electron spins ($\mu = \mu_B$)

$H_{\text{sat}} = 1600T$ for nuclear spins ($\mu = \mu_N = 0.7 \cdot 10^{-3} \mu_B$)

Curie susceptibility: ($H \ll H_{\text{sat}}$)

$$\chi_C = \frac{n\mu^2}{3kT}$$

Usually, if the atoms (molecules) have the free magnetic moments, $|\chi_C| \gg |\chi_L|, |\chi_{VV}|$

E. Magnetism of delocalized electrons. (Metals)

1. Pauli paramagnetism

Origin: Polarization of spins of the free electrons by the external magnetic field.

where the Fermi energy, $\varepsilon_F = \hbar^2(3\pi^2n)^{2/3}/2m$

Variation with temperature: temperature independent

2. Landau diamagnetism

Origin: Orbital movement of free electrons related with Landau quantization.

Variation with temperature: temperature independent.

For the simplest metals $|\chi_P| \gg |\chi_{\text{Landau}}|$

IX. PHASE TRANSITIONS

A. Classification of phase transitions

1. - according thermodynamics

Definition: Phase transition is characterized by anomalies in the macroscopic or thermodynamical properties of the system

I-order = discontinuous (e.g. melting)
-characterized by the latent heat

II-order = **continuous** (most of magnetic transitions)

-there is no latent heat, but other thermodynamical properties (e.g. specific heat) reveals the anomalies

2. - according symmetry

Definition: Phase transition is characterized by qualitative changes, that is, in most the cases is due to the change of the *symmetry* of the system

I. Nonsymmetric (gas-liquid) - always I order

II. Symmetric I or II order

1. Continuous-Continuous (Isotropic liquid - Nematic)

2. Continuous-Discrete (Liquid - crystal)

3. Discrete-Discrete (Structural phase transitions in crystals)

3.a No group-subgroup relations (e.g. graphite-diamond) = reconstructive transitions

- theory can be done only in several cases

3.b With group-subgroup relations

-theory can be done in most the cases

Theory prediction:	Experiment:
I order	I order
II order	I or II order

C. Types of transitions in solids:

1. Magnetic transitions

- **Ferromagnetic:** (spontaneous magnetization \mathbf{M})
- **Antiferromagnetic** (modulation of microscopic magnetization $\mathbf{m}(r)$, can be detected by neutron scattering)
- **More complicated** (Ferrimagnetic, Helico-magnetic etc)

In general:

- If transition is of the I order, theory can be done by chance
- If transition is of the II order, theory can be done in most the cases

B. Thermodynamics of the phase transitions

1. I-order

Characteristic properties: Latent heat q and Hysteresis (supercooling or superheating), discontinuity of all the physical quantities (S, V etc)

Clausius relation:

$$\frac{dT_c}{dp} = T \frac{\Delta V}{q}$$

2. II-order

$$\Delta V = 0, \quad \Delta S = 0$$

Keesom-Ehrenfest relations (1993)

$$\frac{dT_c}{dp} = \frac{\Delta k_T}{\Delta \alpha_p} = \frac{V}{T} \frac{\Delta \alpha_p}{\Delta c_p}$$

where:

-compressibility

$$k_T = -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_T$$

-thermal expansion

$$\alpha_p = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_p$$

-specific heat

$$c_p = T \left(\frac{\partial S}{\partial T} \right)_p$$

Important: all these quantities are measurable experimentally

2. Structural transitions

can be of the **displacive** or of the **order-disorder** type

a. *Ferroc transition* characterized by spontaneous *macroscopic* quantity:

- **Ferroelectric** (spontaneous polarization \mathbf{P})
- **Ferroelastic** (spontaneous deformation)

b. *Modulated (non ferroic) transitions* - **Commensurate** (antiferroelectric)

- **Incommensurate**

D. Landau theory of the II-order transitions

There are two types of theoretical approaches to the phase transitions:

- **Microscopical** (e.g. - Ising model)
- **Phenomenological** - Landau theory

The principal hypothesis of the Landau theory: phase transition is characterized by appearance of the new quantity - **the order parameter** η (e.g. magnetization). Therefore below transition, the symmetry $\eta \rightarrow -\eta$ is broken. The thermodynamic potential, Φ can be expanded in Taylor series over η near transition temperature T_c and $\Phi(\eta) = \Phi(-\eta)$. T, p are considered as independent thermodynamical variables.

$$\Phi(\eta) = \Phi_0 + a(T - T_c)\eta^2 + B\eta^4 + \dots$$

To find η , one should minimize $\Phi(\eta)$:

$$0 = \frac{\partial \Phi}{\partial \eta} = 2a(T - T_c)\eta + 4B\eta^3$$

If $T > T_c$,

$$\eta = 0; \quad \Phi = \Phi_0$$

If $T < T_c$

$$\eta = \pm(-a(T - T_c)/2B)^{1/2}$$

and

$$\Phi = \Phi_0 - \frac{a^2}{4B}(T - T_c)^2$$

The entropy at $T < T_c$

$$S = - \left(\frac{\partial \Phi}{\partial T} \right)_p = S_0 + \frac{a^2}{2B}(T - T_c)$$

and specific heat

$$c_p = T \left(\frac{\partial S}{\partial T} \right)_p = c_{p0} + \frac{a^2}{2B}$$

Specific heat jump:

$$\Delta c_p = \frac{a^2}{2B}$$

If the transition is *ferroelectric* and order parameter makes a sense of polarization ($\eta = P$), the Landau expansion in the electric field E is written as:

$$\Phi(\eta) = \Phi_0 - \eta E + a(T - T_c)\eta^2 + B\eta^4$$

at $T > T_c$ the amplitude of η is small and the term $B\eta^4$ can be neglected. Then, the order parameter above T_c is found from:

$$0 = \frac{\partial \Phi}{\partial \eta} = -E + 2a(T - T_c)\eta$$

$$\eta = \frac{E}{2a(T - T_c)} = \chi E$$

The susceptibility χ satisfies the Curie-Weiss law:

$$\chi = \frac{C_{cw}}{T - T_c}$$

with

$$C_{cw} = \frac{1}{2a}$$

There is an evident relation between the spontaneous polarization below transition:

$$P^2 = -\frac{a(T - T_c)}{2B},$$

jump of specific heat Δc_p at the transition and the Curie-Weiss constant C_{cw} above transition that follows from Landau theory:

$$\frac{P^2}{(T_c - T)} = 2C_{cw} \cdot \Delta C$$

A. Ising Hamiltonian:

Interaction between N spins, located on the one-dimensional periodic lattice is given by the Ising model Hamiltonian:

$$H = -\frac{1}{2}J \sum S_i S_{i+1} - \mu H \sum S_i$$

Where $S_i = \pm 1$, and the ferromagnetic interaction is approximated by positive coupling constant J between nearest neighbors..

In the high-temperature **paramagnetic phase** at $H = 0$ the number of positive spins (directed "up") - N_\uparrow is equal to the number of negative spins (directed "down") - $N_\downarrow = N - N_\uparrow$ and the order parameter,

$$\sigma = \frac{N_\uparrow - N_\downarrow}{N}$$

is equal to zero.

Note, that:

$$N_\uparrow = \frac{N}{2}(1 + \sigma), \quad N_\downarrow = \frac{N}{2}(1 - \sigma)$$

In the low temperature **ferromagnetic phase** at $H = 0$ the order parameter is not equal to zero and the system reveals the **spontaneous magnetization**:

$$M = N\mu\sigma$$

B. Mean field solution of the Ising model

The internal (orientational) energy:

$$U = J(N_{\uparrow\downarrow} + N_{\downarrow\uparrow} - N_{\uparrow\uparrow} - N_{\downarrow\downarrow}) - \mu H(N_\uparrow - N_\downarrow)$$

where $N_{\uparrow\uparrow}$, $N_{\downarrow\downarrow}$, $N_{\downarrow\uparrow}$ are the No of "up-up", "down-down" and "up-down" spin bounds.

Neglecting the correlations between nearest neighbors (**mean field approximation**), we have:

$$N_{\uparrow\uparrow} \approx \frac{1}{2}N_\uparrow \cdot \frac{N_\uparrow}{N} = \frac{N}{8}(1 + \sigma)^2$$

$$N_{\downarrow\downarrow} \approx \frac{1}{2}N_\downarrow \cdot \frac{N_\downarrow}{N} = \frac{N}{8}(1 - \sigma)^2$$

$$N_{\uparrow\downarrow} = N_{\downarrow\uparrow} \approx \frac{1}{2}N_\downarrow \cdot \frac{N_\uparrow}{N} = \frac{N}{8}(1 - \sigma^2)$$

and:

$$U = -J\frac{N}{2}\sigma^2 - \mu H N \sigma$$

The entropy:

$$\begin{aligned}
S &= k \ln \Gamma = k \ln \frac{N!}{N_\uparrow! N_\downarrow!} \\
&= kN \ln N - kN_\uparrow \ln N_\uparrow - kN_\downarrow \ln N_\downarrow \\
&= -k \frac{N}{2} \left[(1+\sigma) \ln \frac{(1+\sigma)}{2} + (1-\sigma) \ln \frac{(1-\sigma)}{2} \right] \\
&= -k \frac{N}{2} \left[(1+\sigma) \ln(1+\sigma) + (1-\sigma) \ln(1-\sigma) \right] + kN \ln 2
\end{aligned}$$

(The Stirling formula

$$N! \approx N^N e^{-N} \quad (N \rightarrow \infty)$$

was used)

The free energy:

$$\begin{aligned}
F &= U - TS \\
&= -J \frac{N}{2} \sigma^2 - \mu H N \sigma - kT N \ln 2 \\
&\quad + kT \frac{N}{2} \left[(1+\sigma) \ln(1+\sigma) + (1-\sigma) \ln(1-\sigma) \right]
\end{aligned}$$

In the thermal equilibrium,

$$\frac{\partial F}{\partial \sigma} = -N \left[J\sigma + \mu H - \frac{kT}{2} \ln \frac{1+\sigma}{1-\sigma} \right] = 0$$

After simple transformations we obtain the **self-consistence equation** for σ :

$$\sigma = \tanh \frac{J\sigma + \mu H}{kT} = \tanh \left(\frac{T_c}{T} \sigma + \frac{\mu H}{kT} \right)$$

where the **critical temperature** $T_c = J/k$.

For $H = 0$ and for $T > T_c$ the unique solution is the **paramagnetic phase** with $\sigma = 0$

For $H = 0$ and for $T < T_c$ there are three solution: one is unstable with $\sigma = 0$ (the paramagnetic phase),

and two degenerate stable solutions with $\sigma = \pm |\sigma_0|$ that corresponds to the **ferromagnetic phase**.

C. Properties of the paramagnetic phase

Above T_c , the order parameter σ vanishes when $H \rightarrow 0$. To find $\sigma(H)$ at small H we linearize the selfconsistence equation with respect to $kT_c + \mu H$:

$$\sigma \approx \frac{T_c}{T} \sigma + \frac{\mu H}{kT}$$

Then,

$$\sigma = \frac{\mu H}{k(T - T_c)}$$

Since $M = N\mu\sigma$ we obtain the **Curie-Weiss paramagnetic susceptibility** for the system of interacting spins that diverges at the transition point:

$$\chi = \frac{\mu^2}{k(T - T_c)} N$$

Remark: for the **antiferromagnetic** phase transition, the **Curie temperature** T_c is negative and is obtained by extrapolation of $\chi^{-1}(T)$ to the region of negative temperatures. The transition to the antiferromagnetic state occurs however at positive **Neel** temperature.

D. Properties of the ferromagnetic phase

When $T \rightarrow 0$ the **order parameter**, $\sigma \rightarrow 1$

Near the **phase transition**, when $T \rightarrow T_c$, the amplitude of σ is small, and the free energy can be expanded in the Taylor series:

$$F/N \approx k \frac{1}{2} (T - T_c) \sigma^2 + \frac{1}{12} k T_c \sigma^4$$

For $T > T_c$ there is only one paramagnetic minimum $\sigma = 0$ and $F = 0$

For $T < T_c$ there are two degenerate minima:

$$\sigma = \pm \sqrt{3} (T/T_c - 1)^{1/2}$$

The free energy depends on the temperature as:

$$F/N \approx -\frac{3}{4} k T_c (T/T_c - 1)^2$$

The entropy below transition:

$$S/N = -\frac{\partial F}{\partial T} = \frac{3}{2} k (T/T_c - 1)$$

The **jump of the specific heat** at the transition:

$$\Delta C/N = T \frac{\partial S}{\partial T} = \frac{3}{2} k$$

The entropy change during the transition is given by $kTN[(1+\sigma) \ln(1+\sigma) + (1-\sigma) \ln(1-\sigma)]/2$ when σ changes from 0 to 1. Finally we have (per one mole):

$$\Delta S = R \ln 2$$

Experimentally, it can be measured as

$$\int_0^{T_c} \frac{C(T) - C_0(T)}{T} dT$$

E. Comparison with exact results:

More sophisticated consideration shows that paraferromagnetic transition occurs only in 2- and 3-dimensional space. The mean field consideration can be considered as satisfactory only at $d = 3$.

There is no phase transition in 1d space (Landau theorem). The exact solution of the Ising model for $d = 1$ gives the following expression for specific heat ($H = 0$):

$$C = \frac{N}{2} \left(\frac{J/T}{\cosh(J/T)} \right)^2$$

That have a maximum at $T \approx 0.8J$

The induced magnetization is given by:

$$M = N\mu \frac{\sinh(\mu H/kT)}{\sqrt{\sinh^2(\mu H/kT) + \exp(-2J/kT)}}$$

XI. MAXWELL'S EQUATIONS OF CONTINUOUS MEDIA

A. Macroscopic Maxwell's equations

The average (macroscopic) field:

Electric: $\mathbf{E} = \langle \mathbf{e} \rangle$, Magnetic: $\mathbf{B} = \langle \mathbf{h} \rangle$

Macroscopic Maxwell's equations:

$$\begin{aligned} \operatorname{div} \mathbf{D} &= 4\pi \rho_{ext} & \mathbf{D} &= \mathbf{E} + 4\pi \mathbf{P} \\ \operatorname{rot} \mathbf{H} &= \frac{1}{c} \frac{\partial}{\partial t} \mathbf{D} + \frac{4\pi}{c} \mathbf{j}_{ext} & \mathbf{H} &= \mathbf{B} - 4\pi \mathbf{M} \\ \operatorname{div} \mathbf{B} &= 0 & \langle \rho \rangle &= \operatorname{div} \mathbf{P} \\ \operatorname{rot} \mathbf{E} &= -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} & \langle \mathbf{j} \rangle &= \partial \mathbf{P} / \partial t + \operatorname{crot} \mathbf{M} \end{aligned}$$

B. Applications of Maxwell's equations in particular cases:

-Isotropic media

-Dispersionless of the dielectric constant and magnetic susceptibility

-low frequencies: $\omega < c/a$, (a is the size of the body) $\omega < 1/\tau$ where τ is the characteristic time: relaxation for dielectrics or mean free path time for a metal.

-In many cases one can assume $\mu \approx 1$

In dielectric:

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 0, & \operatorname{rot} \mathbf{E} &= -\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t}, \\ \operatorname{div} \mathbf{H} &= 0, & \operatorname{rot} \mathbf{H} &= \frac{\varepsilon}{c} \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

Wave equations in dielectric:

$$\Delta \mathbf{H} - \frac{\varepsilon \mu}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad \Delta \mathbf{E} - \frac{\varepsilon \mu}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

The light velocity in dielectric: $c_d = c/\sqrt{\varepsilon \mu} \approx c/\sqrt{\varepsilon}$

In metal

with conductivity σ ($\mathbf{j} = \sigma \mathbf{E}$)

$$\operatorname{div} \mathbf{E} = 0, \quad \operatorname{rot} \mathbf{E} = -\frac{\mu}{c} \frac{\partial \mathbf{H}}{\partial t},$$

$$\operatorname{div} \mathbf{H} = 0, \quad \operatorname{rot} \mathbf{H} = \frac{4\pi}{c} \sigma \mathbf{E}$$

Damping field equations in metal:

$$\Delta \mathbf{H} = \frac{4\pi \sigma}{c^2} \frac{\partial \mathbf{H}}{\partial t} \quad \Delta \mathbf{E} = \frac{4\pi \sigma}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

Damping (Lentz) time and penetration length:

$$\tau = \frac{4\pi \sigma l^2}{c^2} \quad \delta = \sqrt{\frac{c^2}{2\pi \sigma \omega}}$$

Where l is the size of the sample

XII. METAL IN THE HIGH-FREQUENCY ELECTROMAGNETIC FIELD

A. Screening of the h.f. magnetic field

*Problem *:* The metallic plate, having width $2a$, surface S and resistivity σ is placed in the alternating magnetic field $\mathbf{H}_0(t) = \hat{\mathbf{z}} \operatorname{Re} H_0 \omega e^{i\omega t}$ that is parallel to its surface. Find the magnetic $H(x, t)$ and electric fields $E(x, t)$ and electric current $j(x, t)$ inside the plate and the complex magnetic polarizability $\alpha(\omega) = \alpha'(\omega) + i\alpha''(\omega)$ of the plate. Discuss two limit cases when: $a \gg \delta$ and when $a \ll \delta$, where

$$\delta = \sqrt{c^2/2\pi \sigma \omega}$$

is the penetration length. Up to what thickness d_{\min} the results are valid?

Solution:

1. Electromagnetic fields

Magnetic field inside of the plate oscillates with the same frequency ω and depends only on the perpendicular to the plate surface coordinate x , and is directed along $\hat{\mathbf{z}}$. Substituting

$$H(t, x) = H_\omega(x) e^{-i\omega t}$$

into the field equation:

$$\Delta \mathbf{H} = \frac{4\pi \sigma}{c^2} \frac{\partial \mathbf{H}}{\partial t}$$

one obtains:

$$\frac{\partial^2 H_\omega}{\partial a^2} + k^2 H_\omega = 0$$

where

$$k = \sqrt{\frac{4\pi\sigma\omega}{c^2}} i = \frac{(1+i)}{\delta} = \frac{\sqrt{2}}{\delta} e^{i\pi/4}$$

Solution of this equation is given by superposition of two exponents:

$$H_\omega = \frac{1}{2} H_\pm \exp(\pm ikx);$$

because of the $x \rightarrow -x$ symmetry of the system, $H_+ = H_- = H_1$ and

$$H_\omega = H_1 \cos kx.$$

Constant H_1 is found from the boundary conditions:

$$H_\omega(\pm a) = H_{0\omega}$$

Finally we have, $H_1 = H_{0\omega} / \cos ka$ and

$$H_\omega = H_{0\omega} \frac{\cos kx}{\cos ka}.$$

The electric current $\mathbf{j} = \mathbf{j}_\omega e^{-i\omega t}$ is found from: $\text{rot}\mathbf{H} = 4\pi\mathbf{j}/c$:

$$\mathbf{j}_\omega = \mathbf{n}_y H_0 \frac{kc \sin kx}{4\pi \cos ka} = \mathbf{n}_y H_0 \sqrt{\frac{\omega\sigma}{4\pi}} \frac{\sin kx}{\cos ka} e^{i\pi/4}$$

Finally, **the Electric field** is: $\mathbf{E} = \mathbf{E}_\omega e^{-i\omega t}$, where

$$\mathbf{E}_\omega = \mathbf{j}_\omega / \sigma = \mathbf{n}_y H_0 \sqrt{\frac{\omega}{4\pi\sigma}} \frac{\sin kx}{\cos ka} e^{i\pi/4}$$

The phase shift between H and E is equal to $\pi/4$

In the limit case of semi-space, $a \rightarrow \infty$

$$H_y = H_0 e^{-x/\delta} \cos\left(\frac{x}{\delta} - \omega t\right)$$

$$E_x = H_0 \sqrt{\frac{\omega}{4\pi\sigma}} e^{-x/\delta} \cos\left(\frac{x}{\delta} - \omega t - \frac{\pi}{4}\right)$$

The Surface impedance of conductor, ζ is defined as:

$$\mathbf{E}_t = \zeta [\mathbf{H}_t \mathbf{n}_x]$$

In the case of semi-space,

$$\zeta = \sqrt{\frac{\omega}{8\pi\sigma}} (1 - i)$$

The **Magnetic moment** $\mathbf{m} = \mathbf{m}_\omega e^{-i\omega t}$ is calculated as:

$$\begin{aligned} \mathbf{m}_\omega &= \frac{1}{2c} \int_{-a}^a [\mathbf{r} \mathbf{j}_\omega] dV \\ &= \mathbf{n}_z H_0 \frac{kS}{4\pi} \int_0^a \frac{\sin kx}{\cos ka} x dx \\ &= -\mathbf{n}_z H_0 \frac{Sa}{4\pi} \left(1 - \frac{\tan ka}{ka}\right) \end{aligned}$$

therefore, the magnetic polarizability:

$$\alpha(\omega) = \frac{1}{V} \frac{M_\omega}{H_{0\omega}}$$

is written as:

$$\alpha(\omega) = -\frac{1}{4\pi} \left(1 - \frac{\tan ka}{ka}\right)$$

Properties of polarizability:

$$\alpha(\omega) = \alpha'(\omega) + i\alpha''(\omega)$$

Change of the internal energy per unit time:

$$dU = -\mathbf{m} \frac{d\mathbf{H}_0}{dt}$$

Dissipation of the energy (per unit time):

$$\begin{aligned} Q &= \langle -\mathbf{m} \frac{d\mathbf{H}_0}{dt} \rangle \\ &= -\langle \frac{1}{2} (m_\omega e^{-i\omega t} + m_\omega^* e^{i\omega t}) \frac{d}{dt} \frac{1}{2} H_0 (e^{-i\omega t} + e^{i\omega t}) \rangle \\ &= \frac{i\omega}{4} H_0^2 V \langle (\alpha(\omega) e^{-i\omega t} + \alpha^*(\omega) e^{i\omega t}) (e^{-i\omega t} - e^{i\omega t}) \rangle \\ &= -\frac{i\omega}{4} H_0^2 V [\alpha(\omega) - \alpha^*(\omega)] = H_0^2 V \frac{\omega}{2} \alpha''(\omega) \end{aligned}$$

where the averaging is taken over the oscillation period. Therefore, *The dissipation of energy is given by the imagine part of $\alpha(\omega)$.*

Limit cases:

For low frequencies: ($\delta > a$),

$$\begin{aligned} \frac{1}{ka} \tan ka &\approx \frac{1}{ka} (ka + \frac{1}{3}(ka)^3 + \frac{2}{15}(ka)^5) \\ &= 1 + \frac{2}{3}i \left(\frac{a}{\delta}\right)^2 - \frac{8}{15} \left(\frac{a}{\delta}\right)^4 \end{aligned}$$

And, finally:

$$\alpha' = -\frac{2}{15\pi} \left(\frac{a}{\delta}\right)^4 = -\frac{8\pi \sigma^2 \omega^2}{15 c^4} a^4;$$

$$\alpha'' = \frac{1}{6\pi} \left(\frac{a}{\delta}\right)^2 = \frac{1}{3} \frac{\sigma \omega}{c^2} a^2$$

For high frequencies: ($\delta < l$)

$$\frac{1}{ka} \tan ka \approx \frac{1}{ka} = \frac{\delta}{2a} (1 - i)$$

$$\alpha' = -\frac{1}{4\pi} \left(1 - \frac{\delta}{2a}\right) = -\frac{1}{4\pi} \left(1 - \frac{c}{a\sqrt{8\pi\sigma\omega}}\right)$$

$$\alpha'' = \frac{1}{4\pi} \frac{\delta}{2a} = \frac{1}{4\pi a} \frac{c}{\sqrt{8\pi\sigma\omega}}$$

IMPORTANT: The expressions for $\alpha(\omega)$ for the small powder particles have the same structure but with different numerical coefficients. From the measurements of frequency dependence of α' , α'' one can extract the relative value of resistivity σ .

B. Skin effect

*Problem:** The metallic plate of width $2a$ and of resistivity σ is placed in the alternating magnetic field $E(t) = ReE_{0\omega}e^{i\omega t}$ that is parallel to its surface S . Find: electric field $E(x, t)$, electric current $j(x, t)$ and the magnetic field $H(x, t)$ inside the plate.

Find complex resistance R_ω , defined as: $I_\omega = R_\omega^{-1}E_{0\omega}$ (where I_ω is the integral current over all the section of the plate) in two limit cases: $a \gg \delta$ and $a \ll \delta$ where δ is the *Skin depth* $(c^2/2\pi\sigma\omega)^{1/2}$ (=penetration depth of magnetic field)

XIII. DISPERSION OF THE DIELECTRIC CONSTANT

A. The dispersion of the dielectric constant

Material equations:

$$\mathbf{D}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) + \int_0^\infty f(t')\mathbf{E}(\mathbf{r}, t - t')dt'$$

$$\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega); \quad \varepsilon(\omega) = 1 + \int_0^\infty f(\tau)e^{i\omega\tau}d\tau$$

Properties of $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$

$$\varepsilon^*(\omega) = \varepsilon(-\omega)$$

$$\omega\varepsilon''(\omega) > 0$$

Then, $\varepsilon'(\omega)$ is the even function: $\varepsilon'(-\omega) = \varepsilon'(\omega)$ and $\varepsilon''(\omega)$ is the odd function: $\varepsilon''(-\omega) = -\varepsilon''(\omega)$

The absorbed energy in dispersive media

$$Q = \frac{\omega}{4\pi} (\varepsilon''(\omega)\mathbf{E}^2 + \mu''(\omega)\mathbf{H}^2)$$

B. Dielectric constant $\varepsilon(\omega)$ for the quasielastic dipoles

1. The model of quasielastic dipoles:

$$\ddot{d} + \omega_0^2 d + \gamma \dot{d} = \frac{e^2}{m} E(t)$$

$d = er$ is the dipolar moment. For $E(t) = E_\omega e^{-i\omega t}$ we have:

$$d_\omega = \frac{e^2}{m} \frac{E_\omega}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

Taking $\varepsilon(\omega) = (E_\omega + 4\pi P_\omega)/E_\omega$ with $P_\omega = Nd_\omega$, we obtain:

$$\varepsilon(\omega) = 1 + \frac{\Omega^2}{\omega_0^2 - \omega^2 - i\gamma\omega} = \varepsilon'(\omega) + i\varepsilon''(\omega)$$

$$\varepsilon'(\omega) = 1 + \frac{\Omega^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$\varepsilon''(\omega) = \frac{\Omega^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

where $\Omega^2 = 4\pi Ne^2/m$.

$\varepsilon'(\omega)$ is the even function of ω and $\varepsilon''(\omega)$ is the odd function of ω

2. Limit cases of frequencies

a) Small frequencies $\omega \rightarrow 0$:

$$\varepsilon'(\omega) = 1 + \Omega^2/\omega_0^2; \quad \varepsilon''(\omega) = \Omega^2\gamma\omega/\omega_0^4$$

b) Large frequencies $\omega \rightarrow \infty$:

$$\varepsilon'(\omega) = 1 - \Omega^2/\omega^2; \quad \varepsilon''(\omega) = \Omega^2\gamma/\omega^3$$

c) Resonant region: $\omega \sim \omega_0$, small damping $\gamma \ll 2\omega_0$

$$\varepsilon'(\omega) = 1 + \frac{\Omega^2}{2\omega_0} \frac{\omega_0 - \omega}{(\omega_0 - \omega)^2 + (\gamma/2)^2};$$

$$\varepsilon''(\omega) = \frac{\Omega^2}{2\omega_0} \frac{\gamma/2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}$$

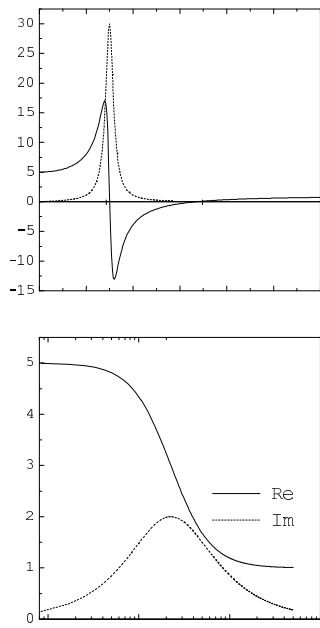


FIG. 1. Real and Imaginary parts of the dielectric constant ϵ for the gas of quasielastic dipoles (a), for the gas of the overdamped dipoles (b) (Debye behaviour) and for the Drude metal (c)

3. Limit cases of damping

Oscillator without damping: ($\gamma \rightarrow 0$)

$$\epsilon'(\omega) = 1 - \frac{\Omega^2}{\omega^2 - \omega_0^2} \quad \epsilon''(\omega) = \pi\Omega^2 \frac{\omega}{\omega_0} \delta(\omega^2 - \omega_0^2)$$

Overdamped oscillator: $\gamma > 2\omega_0$

When $\gamma \rightarrow \infty$

$$\epsilon(\omega) = 1 + \frac{\Omega^2}{\omega_0^2} \frac{1}{1 - i\omega\tau_D}$$

and:

$$\epsilon'(\omega) = 1 + \frac{\Omega^2}{\omega_0^2} \frac{1}{1 + (\omega\tau_D)^2}; \quad \epsilon''(\omega) = \frac{\Omega^2}{\omega_0^2} \frac{\omega\tau}{1 + (\omega\tau_D)^2}$$

Where $\tau = \gamma_D/\omega_0^2$ is the **Debye relaxation time**

The memory function: $\tilde{\omega} \rightarrow i\gamma/2, \sin \tilde{\omega}t \rightarrow i \sinh \gamma/2$

$$f(t) = N \frac{e^2}{\gamma m} (1 - e^{-\gamma t})$$

4. *Free electrons in metal:* $\omega_0 = 0$ (Drude approximation)

Electrons are considered as charged particle in a viscous media with equation of motion:

$$\ddot{r} + \gamma \dot{r} = \frac{e}{m} E(t)$$

The electrical current in a constant field E : $j = Ne \dot{r} = Ne^2 E/m\gamma$ is defined by **the static conductivity**:

$$\sigma = \frac{Ne^2}{m\gamma} = \frac{\Omega^2}{4\pi\gamma}$$

where **the plasma frequency**:

$$\Omega^2 = 4\pi \frac{Ne^2}{m}$$

the characteristic **relaxation time** of the stabilization of the current is:

$$\tau = 1/\gamma$$

In these terms, **the dielectric constants** are written as:

$$\epsilon'(\omega) = 1 - \frac{4\pi\sigma\tau}{1 + (\omega\tau)^2} \quad \epsilon''(\omega) = \frac{4\pi\sigma}{\omega} \frac{1}{1 + (\omega\tau)^2}$$

In the alkali metals the relaxation time $\tau = 1/\gamma \sim 10^{-13} - 10^{-14} s$,

The plasma frequency $\Omega \sim 10^{15} s^{-1}$ corresponds to ultraviolet wave $\lambda = 2\pi c/\Omega \sim 10^{-5} cm \sim 1000 \text{ \AA}$

Therefore $4\pi\sigma/\Omega = \Omega\tau = \Omega/\gamma \gg 1$

The asymptotes at this condition are:

1) $\omega > \gamma$

$$\epsilon'(\omega) = 1 - \frac{\Omega^2}{\omega^2} \quad \epsilon''(\omega) = \frac{\Omega^2\gamma}{\omega^3}$$

2) $\omega < \gamma$

$$\epsilon'(\omega) = 1 - 4\pi\sigma\tau \quad \epsilon''(\omega) = \frac{4\pi\sigma}{\omega}$$

Note, that:

a) $\epsilon'(\Omega) = 0$; $\epsilon'(\omega) > 0$ when $\omega > \Omega$; $\epsilon'(\omega) < 0$ when $\omega < \Omega$

b) $|\epsilon'| > |\epsilon''|$ when $\omega > \gamma$; $|\epsilon'| < |\epsilon''|$ when $\omega < \gamma$

XIV. ANALYTICAL PROPERTIES OF $\varepsilon(\omega)$

ω is considered as a complex value $\omega = \omega' + i\omega''$, then $\varepsilon^*(\omega) = \varepsilon(-\omega^*)$ and, in particular $\varepsilon(i\omega'') = \varepsilon^*(i\omega'')$

$$\begin{aligned} \text{Im}\varepsilon &> 0 & \text{for } \omega &= \omega' > 0 \\ \text{Im}\varepsilon &< 0 & \text{for } \omega &= \omega' < 0 \end{aligned}$$

A. Kramers-Kronig relations:

$$\begin{aligned} \varepsilon'(\omega) - 1 &= \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon''(z)}{z - \omega} dz = \frac{2}{\pi} \mathcal{P} \int_0^{\infty} \frac{z\varepsilon''(z)}{z^2 - \omega^2} dz \\ \varepsilon''(\omega) &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon'(z) - 1}{z - \omega} dz = -\frac{2\omega}{\pi} \mathcal{P} \int_0^{\infty} \frac{\varepsilon'(z)}{z^2 - \omega^2} dz \end{aligned}$$

B. Example: gas of the quasielastic dipoles

$$\begin{aligned} \varepsilon''(\omega) &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\varepsilon'(z) - 1}{z - \omega} dz \\ &= -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Omega^2(\omega_0^2 - z^2)}{(\omega_0^2 - z^2)^2 + \gamma^2 z^2} \frac{dz}{z - \omega} \\ &= -\frac{\Omega^2}{2\pi} \mathcal{P} \int_{-\infty}^{\infty} \left[\frac{1}{\omega_0^2 - z^2 + i\gamma z} + \frac{1}{\omega_0^2 - z^2 - i\gamma z} \right] \frac{dz}{z - \omega} \\ &= \frac{\Omega^2}{2\pi} \mathcal{P} \int_{-\infty}^{\infty} \left[\frac{1}{(z - \omega_1)(z - \omega_2)} + \frac{1}{(z - \bar{\omega}_1)(z - \bar{\omega}_2)} \right] \frac{dz}{z - \omega} \\ &= -\frac{\Omega^2}{2\pi} 2\pi i [\text{Res}_{\omega_1} + \text{Res}_{\omega_2} + \frac{1}{2} \text{Res}_{\omega}] \\ &= -i \left[\frac{\Omega^2}{(\omega_1 - \omega)(\omega_1 - \omega_2)} + \frac{\Omega^2}{(\omega_2 - \omega)(\omega_2 - \omega_1)} \right. \\ &\quad \left. + \frac{\Omega^2}{2(\omega - \omega_1)(\omega - \omega_2)} + \frac{\Omega^2}{2(\omega - \bar{\omega}_1)(\omega - \bar{\omega}_2)} \right] \\ &= i \left[\frac{\Omega^2}{2(\omega - \omega_1)(\omega - \omega_2)} - \frac{\Omega^2}{2(\omega - \bar{\omega}_1)(\omega - \bar{\omega}_2)} \right] \\ &= \frac{\Omega^2 \gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \end{aligned}$$

The contour integral was taken over the lower half-plane where the poles $\omega_{1,2} = \pm \sqrt{\omega_0^2 - (\gamma/2)^2} - i\gamma/2$ are located. Another pole, $z = \omega$, is located at the real axis. It was passed from the below. The relation $0 = \oint = \mathcal{P} \int_{-\infty}^{\infty} + 2\pi i [\text{res}_{\omega_1} + \text{res}_{\omega_2} + \frac{1}{2} \text{res}_{\omega}]$ was used.

C. The memory function for quasi-elastic dipoles

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \kappa(\omega) e^{-i\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{(\varepsilon(\omega) - 1)}{4\pi} d\omega \\ &= \frac{Ne^2}{2\pi m} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t}}{\omega_0^2 - \omega^2 - i\gamma\omega} d\omega = N \frac{e^2}{m} \frac{\sin \tilde{\omega} t}{\tilde{\omega}} e^{-\gamma t/2} \end{aligned}$$

where $\tilde{\omega} = \sqrt{\omega_0^2 - (\gamma/2)^2}$

Details of calculations:

$$\frac{Ne^2}{2\pi m} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t} d\omega}{\omega_0^2 - \omega^2 - i\gamma\omega} = -\frac{Ne^2}{2\pi m} \int_{-\infty}^{+\infty} \frac{e^{-i\omega t} d\omega}{\omega^2 + i\gamma\omega - \omega_0^2}$$

Poles: $\omega^2 + i\gamma\omega - \omega_0^2 = 0 \Rightarrow \omega_{1,2} = \pm \sqrt{\omega_0^2 - (\gamma/2)^2} - i\gamma/2 = \pm \tilde{\omega} - i\gamma/2$

The integral path should be in the lower half-plane.

$$\begin{aligned} f(t) &= \frac{Ne^2}{m} i \sum_{\text{res}_{1,2}} \frac{e^{-i\omega t}}{(\omega - \omega_1)(\omega - \omega_2)} \\ &= \frac{Ne^2}{m} i \left[\frac{e^{-i\omega_1 t}}{\omega_1 - \omega_2} + \frac{e^{-i\omega_2 t}}{\omega_2 - \omega_1} \right] \\ &= \frac{Ne^2}{m} \frac{i}{\omega_1 - \omega_2} [e^{-i\omega_1 t} - e^{-i\omega_2 t}] = N \frac{e^2}{m} \frac{\sin \tilde{\omega} t}{\tilde{\omega}} e^{-\gamma t/2} \end{aligned}$$

XV. PROPAGATION OF ELECTROMAGNETIC WAVES

A. General

Wave equations for $\mathbf{H} = \mathbf{H}_{\omega} e^{-i\omega t}$, $\mathbf{E} = \mathbf{E}_{\omega} e^{-i\omega t}$, ($\mu \sim 1$), $\varepsilon = \varepsilon(\omega)$

$$\Delta \mathbf{H}_{\omega} + \frac{1}{c^2} \varepsilon(\omega) \omega^2 \mathbf{H}_{\omega} = 0 \quad \Delta \mathbf{E}_{\omega} + \frac{1}{c^2} \varepsilon(\omega) \omega^2 \mathbf{E}_{\omega} = 0$$

solution: $\mathbf{H} = \mathbf{H}_{\omega, k} e^{i\mathbf{k}\mathbf{r} - i\omega t}$, $\mathbf{E} = \mathbf{E}_{\omega, k} e^{i\mathbf{k}\mathbf{r} - i\omega t}$

Dispersion equation:

$$kc = \tilde{n}(\omega)\omega$$

Refraction Index:

$$\tilde{n} = \sqrt{\varepsilon} = n + ik$$

$$\tilde{n} = \sqrt{\varepsilon} = \sqrt{n^2 + k^2} e^{i\varphi} \quad \varphi = \text{arctg}(k/n)$$

n - refraction index

k - absorption coefficient (extinction coefficient)

If $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$,

$$\varepsilon' = n^2 - k^2 \quad \varepsilon'' = 2nk$$

or

$$n = \sqrt{\frac{\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}{2}} \quad \kappa = \sqrt{\frac{-\varepsilon' + \sqrt{\varepsilon'^2 + \varepsilon''^2}}{2}}$$

The electromagnetic waves are:

- propagated in the media when: $\varepsilon' > 0$, $\varepsilon'' \ll \varepsilon'$
- reflected from the media when: $\varepsilon' < 0$, $\varepsilon'' \ll \varepsilon'$
- absorbed by the media when: $\varepsilon'' > \varepsilon'$

Boundary conditions for the Incident (*i*) Reflected (*r*) and Propagating (*p*) electromagnetic waves (the wave is perpendicular to surface):

$$E_i + E_r = E_p;$$

$$H_i + H_r = H_p; \implies k_i E_i + k_r E_r = k_p E_p;$$

where

$$k_i = -k_r; \quad k_p = \tilde{n}(\omega)\omega/c$$

The reflection coefficient:

$$R = \frac{\langle E_i^2 \rangle}{\langle E_r^2 \rangle} = \left| \frac{\tilde{n} - 1}{\tilde{n} + 1} \right|^2 = \frac{(n - 1)^2 + \kappa^2}{(n + 1)^2 + \kappa^2}$$

when damping is small ($\kappa \ll n$)

$$R = \frac{(n - 1)^2}{(n + 1)^2}$$

damping of the propagating wave: $I(z) = I_0 e^{-\alpha z}$
since $I \sim E^2$, the damping coefficient

$$\alpha = 2\omega\kappa/c$$

B. Propagation in dielectrics

In the model of quasi-elastic dipoles: (damping, γ , is small),

$$\varepsilon'(\omega) = 1 - \frac{\Omega^2}{\omega^2 - \omega_0^2} \quad \varepsilon''(\omega) = \pi\Omega^2 \frac{\omega}{\omega_0} \delta(\omega^2 - \omega_0^2)$$

The dispersion of the electromagnetic wave, obtained from the equation: $kc = \sqrt{\varepsilon'(\omega)}\omega$ is:

$$\omega^2 = \frac{1}{2}(k^2 c^2 + \omega_0^2 + \Omega^2) \pm \frac{1}{2} \sqrt{(k^2 c^2 + \omega_0^2 + \Omega^2)^2 - 4(k^2 c^2 + \omega_0^2)}$$

The upper branch of the wave is called as **polariton wave**.

k is imagine in the **optical gap**, where the incident wave is reflected, between *Transversal* and *Logitudinal* frequencies of the optical phonons, $\omega_T < \omega_L$ that are defined by:

$$\varepsilon(\omega_T) = \infty, \quad \varepsilon(\omega_L) = 0$$

The Liddan Sacs Teller relation:

$$\frac{\omega_T^2}{\omega_L^2} = \frac{\varepsilon_\infty}{\varepsilon_0}$$

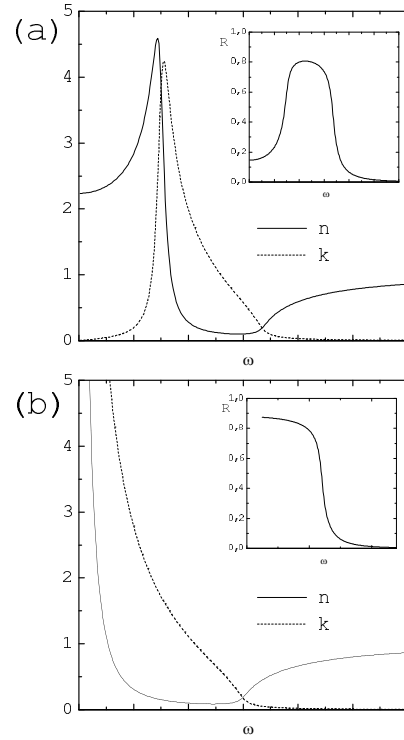


FIG. 2. Refraction Index n and absorption coefficient κ in dielectrics (a) and in metals (b). The inserts shows the reflection coefficient R

Propagation of electromagnetic wave through the dielectric plate

*Problem:** The incident electromagnetic wave is perpendicular to the dielectric plate of width d with dielectric constant $\varepsilon(\omega)$. Find the reflection coefficient $R(\omega)$ Find the propagation coefficient $D(\omega)$. Plot $R(\omega)$, $D(\omega)$ for the model of quasi-elastic dipoles with very small dumping, assuming that $d < \lambda$ (at $\omega = \Omega$)

C. Propagation in metals

In the Drude model the dielectric constant is given by:

$$\varepsilon'(\omega) = 1 - \frac{4\pi\sigma\tau}{1 + (\omega\tau)^2} \quad \varepsilon''(\omega) = \frac{4\pi\sigma}{\omega} \frac{1}{1 + (\omega\tau)^2}$$

One can distinguish three different frequency regions:

1. $\omega < \gamma$ is the **absorption region**. (the radio-frequencies).

When the size of the sample is smaller as the wavelength, the *Quasistatic* approximation is applicable (see

The imaginary part is dominated in a **dielectric constant**

$$\varepsilon(\omega) \approx i\varepsilon''(\omega) = i\frac{4\pi\sigma}{\omega}$$

The **reflection coefficient** is estimated as (Hagen-Rubens relation):

$$R \approx 1 - \left(\frac{2\omega}{\sigma\pi}\right)^{1/2}$$

2. $\gamma < \omega < \Omega$ is the **reflection region** (infrared, usual light)

The **dielectric constant** is given by:

$$\varepsilon'(\omega) = 1 - \frac{\Omega^2}{\omega^2} \quad \varepsilon''(\omega) = \frac{\Omega^2\gamma}{\omega^3}$$

The damping, ε'' , is small and ε' is negative.

3. $\omega > \Omega$ is the **transparency region** (ultraviolet).

The $\varepsilon(\omega)$ is given by the same formulas as in the reflection region. ε' is positive

The **plasma waves (plasmons)** can propagate in this region.

The plasmons **dispersion** is found from $kc = \tilde{n}(\omega)\omega$:

$$\omega(k) = \sqrt{\Omega^2 + k^2c^2}$$

damping of plasmons is as small as $\Omega\tau \sim 10^{-2}$

IMPORTANT: The interband transitions in real metals result to amplification of absorption above the edge frequency (usual light-ultraviolet)

ELECTROMAGNETIC WAVES

λ	ω	Radiation
$10^6 km - 10 km$	$3 \cdot 10^1 - 3 \cdot 10^4 Hz$	Alternating current
$10 km - 1 mm$	$3 \cdot 10^4 - 3 \cdot 10^{11} Hz$	Radio-frequencies
$1 mm - 1 \mu m$	$3 \cdot 10^{11} - 3 \cdot 10^{14} Hz$	Infrared
$\sim 1 \mu m$	$\sim 3 \cdot 10^{14} Hz$	Light
$1 \mu m - 1 nm$	$3 \cdot 10^{14} - 3 \cdot 10^{17} Hz$	Ultraviolet
$1 nm - 5 \cdot 10^2 \text{ \AA}$	$3 \cdot 10^{17} - 5 \cdot 10^{19} Hz$	X-ray
$5 \cdot 10^2 \text{ \AA} - \dots$	$5 \cdot 10^{19} - 10^{23} \dots Hz$	γ -radiation

* $1 cm = 10^4 \mu m = 10^7 nm = 10^8 \text{ \AA}$

The **light wavelength**: 7600 – 6200.Å - Red; 6200 – 5850.Å - Orange; 5850 – 5750.Å - Yellow; 5750 – 5500.Å - Green; 5500 – 4800.Å - Light-Blue; 4800 – 4500.Å - Blue; 4500 – 3800.Å - Violet

XVI. DIELECTRIC CONSTANT IN METAL AND IN NONDEGENERATE PLASMA. (THOMAS-FERMI THEORY)

A. Fermi statistics of electrons

In *plasma* the negative charges of electron gas are compensated by the positive (ions) background.

Fermi distribution function:

$$f(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT} + 1}$$

$\varepsilon = \hbar^2 k^2 / 2m$; $\mu = \mu(T)$ is given by a normalization condition:

$$2 \int f(\varepsilon) \frac{d^3 \mathbf{k}}{(2\pi)^3} = n$$

n - is a concentration of electrons, factor 2 corresponds to the two orientations of spin.

In an external potential $\varphi(r)$, $\varepsilon(k, r) = \hbar^2 k^2 / 2m - e\varphi(r)$ and

$$f(\varepsilon) = \frac{1}{e^{(\hbar^2 k^2 / 2m - \mu - e\varphi(r)) / kT} + 1}$$

Particular cases:

a) **Degenerate plasma** (electrons in a metal) $\mu \gg kT$ at $T = 0$:

$$f(k, r) = \frac{1}{2}[\theta(\hbar^2 k^2 / 2m - \varepsilon_F - e\varphi(r)) + 1];$$

$$\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}; \quad k_F = (3\pi^2 n)^{1/3}; \quad \varepsilon_F = \frac{\hbar^2 (3\pi^2 n)^{2/3}}{2m};$$

b) **Nondegenerate plasma** $\mu \ll kT$

$$f(k, r) = e^{(\hbar^2 k^2 / 2m - \mu - e\varphi(r)) / kT}$$

$$\mu(T) = kT \ln \frac{4\pi^3 \hbar^3 n}{(2\pi m kT)^{3/2}}$$

In normal metal with $n \sim 10^{22} cm^{-3}$: $\varepsilon_F \sim 50.000 K$. The electron gas becomes nondegenerate (at room temperatures) at $n < 10^{19} cm^{-3}$. In semiconductors: $n \sim 10^{17} - 10^{19} cm^{-3}$;

B. Screening

External charge ρ^{ext} in metal is screened by inducing the distribution of internal charge $\rho^{ind}(r)$.

Poisson equation:

$$-\Delta\varphi(r) = 4\pi(\rho^{ext} + \rho^{ind})$$

The principal idea of the screening theory is to assume that $\rho^{ind} = \rho^{ind}(\varphi(r))$ and then, to solve the Poisson equation.

For the Fermi distribution, the induced charge

$$\rho^{ind}(\varphi(r)) = -e[n(\mu + e\varphi(r)) - n(\mu)] = -e^2 \frac{\partial n}{\partial \mu} \varphi(r)$$

The Thomas-Fermi wave vector:

$$k_0^2 \equiv 4\pi e^2 \frac{\partial n}{\partial \mu}$$

Then,

$$-\Delta\varphi(r) + k_0^2\varphi(r) = \rho^{ext}$$

For the point charge $\rho^{ext} = Q\delta(r)$ and the solution of equation:

$$\varphi(r) = \frac{Q}{r} e^{-k_0 r}$$

Particular cases:

a) degenerate electron gas: $\partial n/\partial \mu = \partial n/\partial \varepsilon_F = 3/2\varepsilon_F$.

$$\frac{k_0^2}{k_F^2} = \frac{4}{\pi} \frac{me^2}{\hbar^2 k_F^2} = \left(\frac{16}{3\pi^2}\right)^{3/2} \frac{1}{k_F a_B}$$

Where $a_B = \hbar^2/me^2$ is the Bohr radius.

Hence, the screening length, $2\pi/k_0$, is of the order of 2-6 interatomic distances..

a) nondegenerate plasma (Debye screening) $\partial n/\partial \mu = n/kT$;

$$k_0^2 = 4\pi n e^2 / kT$$

The screening length increases with temperature.

C. Dielectric constant

Dispersion:

$$\mathbf{D}(\mathbf{r}) = \int \varepsilon(\mathbf{r} - \mathbf{r}') \mathbf{E}(\mathbf{r}') d^3 r'$$

For the external (point) charge: $\rho = \rho^{ext} + \rho^{ind}$; $\varphi = \varphi^{ext} + \varphi^{ind}$; $\rho^{ext} = Q\delta(r)$; $\varphi^{ext} = -q/r$

$$\text{div} \mathbf{D} = 4\pi \rho^{ext}; \quad \text{div} \mathbf{E} = 4\pi(\rho^{ext} + \rho^{ind});$$

$$\mathbf{D} = \nabla \varphi^{ext}; \quad \mathbf{E} = \nabla \varphi;$$

Then,

$$\varphi^{ext}(\mathbf{r}) = \int \varepsilon(\mathbf{r} - \mathbf{r}') \varphi(\mathbf{r}') d^3 r'$$

More convenient is to use the Fourier transformation:

$$\varphi^{ext}(\mathbf{q}) = \varepsilon(\mathbf{q}) \varphi(\mathbf{q})$$

or,

$$\varepsilon(\mathbf{q}) = \varphi^{ext}(\mathbf{q}) / \varphi(\mathbf{q})$$

The dependencies, $\varphi^{ext}(\mathbf{q})$, $\varphi(\mathbf{q})$ are found from unscreened and screened Fourier transformed Poisson equations:

$$q^2 \varphi^{ext}(\mathbf{q}) = 4\pi Q$$

$$(q^2 + k_0^2) \varphi^{ext}(\mathbf{q}) = 4\pi Q$$

Finally,

$$\varepsilon(\mathbf{q}) = 1 + \frac{k_0^2}{q^2}$$

Properties:

$\varepsilon(\mathbf{q}) = \infty$ when $q = 0$ (dielectric constant of metal at large scales)

$\varepsilon(\mathbf{q}) = 1$ when $q = \infty$ (dielectric constant of vacuum)